Communication in the Shadow of Catastrophe*

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Abstract

We perform distributional comparative statics in a cheap talk model of adaptation. Receiver borne adaptation costs drive a wedge between the objectives of sender and receiver that is increasing in the magnitude of adaptation. We allow for infinite supports with infinite disagreement at the extremes and compare communication to simple delegation. We find that increases in risk are detrimental to communication: Linear transformations of the state (scale) change communication as well as delegation payoffs proportionally. By contrast, convex transformations of the state (shape) only reduce communication payoffs and make delegation relatively more attractive. Our finding extends to the comparison of distributions with thin (logconcave) versus heavy (logconvex) tails.

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1 Introduction

This paper studies communication between an expert and a decision-maker. Whereas the expert exclusively cares about scientific aspects, the decision-maker also faces a political or firm-specific cost of adapting to the expert's advice. The cost drives a wedge between the expert's and the decision-maker's objectives that is increasing in the magnitude of adaptation. As we allow for infinite supports, there is infinite disagreement at the extreme states. An inadequate response to advice in an extreme state is what we call a *catastrophe* in this paper. Our main focus is on the impact of the distributional environment on communication, in particular, on the relative likelihood of extreme states and extreme disagreement. We wonder how equilibrium advice and equilibrium payoffs change when extremes states get more extreme and more likely. We provide tools in terms of stochastic orders that allow us to rank distributions with respect to their impact on equilibria and payoffs. We compare the communication outcomes to the outcomes under the alternative decision protocol of simple delegation. We find that delegation becomes relatively more attractive when extreme conflicts become more likely.

To illustrate, consider the following examples of communication with catastrophic outcomes: Experts suggested to adjust the drilling procedure in face of new evidence prior to the oil drilling blowout on the *Deepwater Horizon*, *BP* however decided not to change its procedures against expert advice. At the *Challenger* space shuttle explosion engineers warned in vain about potential problems arising from the low temperatures. Officials delayed the evacuation of the *Ahr* valley despite experts' warnings of an extreme rise of the water level and a flooding of the *Ahr* valley. Communication on the brink of these catastrophes was evidently unable to avert them. Expert advice was not sufficiently taken into account. Is it better to delegate the decision rights to the expert when extreme events are possible? Or when such events are relatively likely?

To address our question, we introduce the following model. A decision maker (receiver) seeks advice from a scientific expert (sender). The sender perfectly observes the state of the world and sends a cheap talk message (Crawford and Sobel (1982)) to the receiver who then takes an action. The sender as well as the receiver care about the right action being taken given the state. The decision maker, however, faces additional concerns such as political or firm-specific costs of adaptation. Formally, sender and receiver share a common payoff component aiming at minimizing a common quadratic loss function. In addition to the common loss, the receiver faces a quadratic cost of adaptation; costs are zero if the decision-maker takes the prior optimal action and increase in the deviation of the implemented action from the prior optimal action. As a consequence, the ideal choice functions of sender and receiver are linear with different slopes and intersection at the prior optimal action. There is thus a state dependent bias between sender and receiver that increases in the distance to the expected state, the mean.

We first establish existence and essential uniqueness of cheap talk equilibria that induce a given number of receiver actions up to a countable infinity (Proposition 1). Logconcavity and symmetry of the density in addition to linearity of the bias ensure that our environment features enough regularity to prove these results.

We then focus on linear transformations of the state variable. We fix the shape of the distribution and *scale* the distribution up by increasing the variance. We consider all symmetric distributions with logconcave densities, a rich class of distributions. As an example, imagine the uniform distribution and making the support wider. We show that the result is a linear spread of the equilibrium actions, resulting in a reduction of expected utilities that is proportional to the increase in the variance (Lemma 1). Since the payoff under delegation is also proportional to the variance (Lemma 2), both payoffs are linearly decreasing in the variance. Hence such linear change never implies a switch in the optimal decision-procedure from communication to delegation (Corollary 1).

For the further comparison of distributions, we consider changes that keep *mean and variance* constant. Since variance as a scale variable does not explain the choice of institutions, this is a necessary step. It poses, however, considerable analytical difficulties: standard stochastic comparisons such as a mean preserving spread, the convex order, and a standard monotonic likelihood ratio all imply an increase in variance and are thus ruled out.

We next focus on convex transformations of the state variable. We fix the variance and change the *shape* of the distribution by making the tail of the distribution heavier. Formally, we compare symmetric distributions whose half-distributions (distributions folded at the mean) are ordered in the *convex transform order* (van Zwet (1964)). We show that the quantiles of the sender's equilibrium marginal types are ordered such that all quantiles are larger in the more convex/risky environment (Proposition 2). This result holds irrespective of whether the supports are bounded or unbounded. The convex transform order is invariant to changes in scale, it orders equilibrium strategies in the quantile space. Since the equilibrium strategies are scale-dependent, we need to complement the convex transform order with a scale-dependent order that allows us to order the equilibrium strategies in the state space to get sharper predictions.

We first compare two distributions with bounded supports, such that the more risky one has a larger support. While the half-distributions must violate the monotone likelihood ratio property for such supports, it is still possible to obtain a modified version of the order when equalizing the supports of the distributions. We obtain a clear comparison this way: the equilibrium strategies form a mean-preserving spread *proportional to the size of the support* (Proposition 3). While this comparison of distributions explains everything when it comes to equilibria, the approach is not useful for expected utilities because the magnitude of the scaling factor needed to equalize the supports remains unexplained.

We then compare distributions without correcting for differences in supports, allowing for bounded and unbounded supports. Our comparison is based on the *uniform conditional variability order* (Whitt (1985)). Distributions for which the half-distributions are comparable by the uniform conditional variability order and the convex transform order feature a unimodal likelihood ratio and their cdfs cross once on each side of the prior mean: the less risky distribution is stochastically higher for small deviations from the prior mean, while the more risky distribution is stochastically higher for large deviations from the prior mean. In combination with the ordering of quantiles, we show that this implies that the distribution of receiver actions in the less risky environment is a mean preserving spread of the corresponding actions in the more risky environment for sufficiently high marginal costs of adaptation (Corollary 2). Thus, there is more information transmission in the less risky environment.

To quantify our comparison and the 'sufficiently' high marginal cost of adaptation, we rely on dynamic programming methods. We derive a lower bound on the payoff gains that result from communication (Proposition 5) under the assumption that the distribution features a *convex tail truncated expectation* function – i.e., the conditional expectation, conditional on a truncation to the tail, as function of the truncation point is convex in the truncation point. This implies that the distribution becomes more risky when truncated to the tail. The Gauss distribution is an example. The result generalizes the closed form expression for the gain from communication for *linear tail truncated expectations*, satisfied by the class of generalized Pareto distributions (Deimen and Szalay (2019)).

We then use these bounds and closed form expressions to come back to explaining the connection between risk and the choice between communication and delegation. For the generalized Pareto distribution, we obtain a function – of the shape parameter of the distribution and the alignment of interest parameter – that characterizes indifference between the two institutions (Proposition 6). In more risky environments there is more delegation at the optimum. Comparing the Gauss distribution to the more risky Laplace distribution, we confirm that at the optimum there is more communication in the less risky environment (Proposition 7).

Last but not least, we consider environments with infinite supports that feature thin – sub-exponential – tails versus those that feature heavy – super-exponential – tails. These environments feature half-distributions that are ordered in both the convex transform order and the uniform conditional variability order. Consequently, our result that communication suffers in risky environments extends (Proposition 8). Vice versa, we show that the introduced partial stochastic orders capture the essential comparison of thinner versus heavier tails.

To summarize, linear transformations of random variables – changes in scale – make everyone worse off but don't trigger any changes in the optimal institutions. In contrast, convex transformations of random variables decrease the gains under communication and imply that delegation becomes more often optimal for the receiver. It is somewhat surprising that the receiver prefers to delegate for an increased likelihood of extreme states: from the receiver's perspective, the sender strongly overreacts in extreme states which imposes very high adaptation costs associated with delegation. The problem with keeping authority, however, is that the sender knows that the receiver is reluctant to follow the advice: the receiver takes relatively more cautious actions and therefore the expert exaggerates even more. As a consequence equilibrium partitions are coarsened and the quality of communication suffers substantially. Delegation, in contrast, allows the receiver to commit to following the sender's advice and thus makes better use of the sender's superior information.

The following well-documented example illustrates catastrophic decision making based on communication in risky environments.[1] The oil drilling industry in the Gulf of Mexico faced a drastic change in the production environment between 1990 and 2009 moving from shallow to deep water. During this period the oil production from deepwater wells increased from 4.4% to 80% of the total volume. ("Deepwater energy exploration and production [...] involve risk for which neither industry nor government has been adequately prepared [...]." p.9) *British Petroleum* (*BP*) was drilling for oil from the rig *Deepwater Horizon* in the *Macondo* well in the Gulf of Mexico, when in 2010 a blowout with catastrophic consequences occurred. *BP*, the owner of the drilling rights, relied on a subcontractor, *Transocean*, to perform the drilling. *BP* (the receiver) directed the work, *Transocean* (the sender) provided advice, the drilling rig, and the crew operating it. *BP* and *Transocean* had agreed on a budget and a timeline (p.2). Every adaptation away from the planned procedure was costly to *BP* with costs increasing in the length of the resulting delay. As a consequence,

[1] The information provided here is based on the report of the National Comission on the BP Deepwater Horizon oil spill and offshore drilling (2011).

BP responded conservatively to proposed changes by *Transocean* and made decisions that deviated from the recommended actions (p.125). The report states that "Most, if not all, of the failures at *Macondo* can be traced back to underlying failures of management and communication." (p.122).[2]

We contribute to the literature on adaptation in organizations. Alonso et al. (2008) and Rantakari (2008) investigate whether decision-authority should reside at the top of a hierarchy or further down at the level of division-management. Even though these papers consider multi-divisional organizations in which there are additional concerns of coordination, the optimal allocation of authority essentially resolves a delegation versus communication trade-off. These papers as our's use the communication model with linear state-dependent bias that was first studied by Melumad and Shibano (1991). Imperfect profit sharing in their models and adaptation costs in our model provide a micro foundation for such linear conflicts. Since the adaptation costs are increasing in the size of the adjustment, the wedge between the expert's and the receiver's objective is largest at the extremes of the support. This gives a natural connection to catastrophic outcomes in extreme states and to such outcomes becoming more likely if the state distribution features heavier tails. Our analysis can directly be applied to situations in which the state can a priori only take positive values. Moreover, it can be extended to the Crawford and Sobel (1982) model with a one-sided bias. We leave this to future work.

More recent contributions to the adaptation literature include Rantakari (2013), Dessein et al. (2022), and Liu and Migrow (2022). Liu and Migrow (2022) analyze a model of disclosure with information acquisition. They show that the distribution of an uncertainty parameter has an important impact on the optimal allocation of decision-rights in their problem. Rantakari (2013) allows firms to choose the compensation and the authority structure jointly. He finds that firms that operate in volatile

[2]A catastrophe in this paper corresponds to a catastrophically low payoff that is human made. It stems from large disagreement in exceptional situations. There are other situations in which a catastrophe is commonly anticipated – imagine a meteorite coming earth's way. Such a situation could arguably align interests perfectly. While this is an important and interesting situation, we focus on human made catastrophes in this paper leaving other cases to future work.

environments are characterized by decentralized decision making and a compensation with focus on performance at the division level. Dessein et al. (2022) provide a theoretical model predicting that an environment that is more volatile locally results in more decentralized decision making only when the need for coordination across subunits is low. We bring new tools to this literature which typically focuses on volatility in the sense of an increase in the variance of a uniform state. Because we study the impact of heavier tails on unbounded supports, instead of the usually assumed compact state space, we need to build our model from scratch. We prove existence and uniqueness of equilibria, analyze the role of the variance for all distributions with a fixed shape, and study variations in the shape of distributions. We provide comparative statics on the heaviness of the tails which have not been studied before in the context of strategic communication.

Related cheap talk models with endogenous conflicts are Deimen and Szalay (2019) and Antić and Persico (2020). Antić and Persico (2020) consider various ways in which conflicts can arise endogenously, e.g. trading in a stock market prior to communication in a firm. In Deimen and Szalay (2019) a sender acquires noisy signals about a multidimensional state. Depending on the sender's choice of information, conflicts with the receiver arise. We show that communication is better than delegation in a multidimensional elliptical generalized Pareto environment with heavy tails. Our analysis here builds on our earlier work and provides extensions and generalizations in various directions. We do not impose any functional form on the distribution but use the generalized Pareto distribution as an illustrative example.

The essential new perspective that we bring to the comparison of communication and delegation is the impact of arbitrarily large conflicts. This complements the focus of Dessein (2002), who is the first to study this comparison in the seminal paper of Crawford and Sobel (1982). He shows that whenever interests are sufficiently aligned such that influential communication is possible, the receiver prefers to delegate. When the conflict between sender and receiver gets arbitrarily small, Dessein (2002) shows that payoffs from simple delegation approach first-best payoffs faster than those arising from strategic communication.[3][4] In our setup, increasing the likelihood of extreme conflict has a detrimental impact on communication so that delegation becomes relatively better.

Chen and Gordon (2015) also perform distributional comparative statics in strategic information transmission. They show that information transmission is improved when ideal choices are closer, which is satisfied when the distributions are ordered by the monotone likelihood ratio order. We apply a scaled version of the monotone likelihood ratio order to our analysis on bounded supports. The approach is useful to obtain comparative statics results on equilibrium strategies. It can, however, not be used for comparative statics of expected utilities, because of a scaling factor.

To explain the impact of heavier tails on infinite supports on expected utilities, we go back to other stochastic orders. Whitt (1985)'s uniform conditional variability order and van Zwet (1964)'s convex transform order are long known in statistics (see also Shaked and Shanthikumar (2007)) but have not been studied in the literature on strategic information transmission. In fact, with few exceptions, the economic theory literature has paid little attention to the shape of distributions. Jewitt (2004) offers an insightful overview of problems in which shape matters. He provides connections among partial orders that describe shape, among them van Zwet's convex transform order. More recently, Di Tillio et al. (2021) show that shapes of distributions, measured by the convex transform order, have a decisive effect on whether winning bids contain more or less information than all the bids in an auction.

The remainder of the paper is organized as follows. We present our formal model in Section 2. Equilibria of the communication game are derived in Section 3. This section also studies the impact of linear transformations on equilibria and payoffs. In Section 4, we consider convex transformations. We combine stochastic orders to derive equilibrium and payoff comparisons for different distributions. In Section 5, we

[3]Interestingly, an increase in variance – the support of the distribution – has a similar effect than direct reductions of the sender's bias (effectively, it is the ratio of bias over standard deviation that determines the quality of communication and delegation). Therefore, changes in variance have consequences for the delegation-communication choice in this model.

[4]See Dilmé (2022) for approximate characterizations of strategic information transmission equilibria with small biases.

introduce a dynamic programing method to quantify the gain from communication. We apply our findings to the generalized Pareto distribution and the Gauss distribution. An extension to a comparison of thin versus heavier tailed distributions is given in Section 6. Section 7 concludes. All proofs are in the appendix.

2 Model

We consider a game with two players, a sender S and a receiver R. Sender and receiver have preferences that reflect a common adaptation motive captured by quadratic payoffs that depend on an action $y \in \mathbb{R}$ and on the realization θ of state of the world Θ. For the sender,

$$u_S(y,\theta) = -(y-\theta)^2.$$

The receiver faces an additional cost of adaptation $c(y) = \gamma \cdot y^2$, with $\gamma > 0$ such that $u_R(y,\theta,\gamma) = -(y-\theta)^2 - \gamma \cdot y^2$. Defining $a := \frac{1}{1+\gamma}$, the ideal choice functions of sender and receiver are $y_S(\theta) = \theta$ and $y_R(\theta) = a \cdot \theta$, respectively, where $a \in (0,1)$ as $\gamma > 0$. Because of the additional cost, the receiver adapts more conservatively than the sender. The parameter a measures the *alignment of interests*, with higher values corresponding to more alignment.[5] Since positive affine transformations of utility functions describe the same preferences, we conveniently merge the receiver's motives into one loss function and write[6]

$$u_R(y,\theta,a) = -(y - a\cdot\theta)^2.$$

The state of the world Θ is a random variable with a common prior distribution F with density f on an interval support $\mathcal{S} = \left[\underline{\mathcal{S}}, \overline{\mathcal{S}}\right] \subseteq \mathbb{R}$. We assume that the density is symmetric ($\underline{\mathcal{S}} = -\overline{\mathcal{S}}$), logconcave, and that the mean is zero and the variance σ^2 is finite. Logconcavity ensures that optimal choices and expected utilities are well

[5]In terms of the literature, the sender's bias is state dependent and equals $(1-a)\cdot\theta$.

[6]Note that $u_R(y,\theta,\gamma) = -\frac{1}{a}(y - a\cdot\theta)^2 - (1-a)\cdot\theta^2$, with $\gamma = \frac{1-a}{a}$. The transformation obviously affects levels of utility, but does not impact choices at any margin. The ideal choice functions of the two specifications are the same, and, moreover, the specfications feature the same trade-offs when choosing among institutions of decision-making.

defined and that the tail of the distribution is relatively thin. In particular, we rule out distributions with tails that are heavier than exponential (Laplace). Symmetry of the density implies that we can write $f(\theta) = \kappa \frac{1}{\sigma} \psi\left(\frac{\theta^2}{\sigma^2}\right)$, where κ is a normalizing constant and ψ is a (density generator) function that captures the shape of the distribution.[7] Importantly, the density depends only on the standardized variable $\frac{\theta}{\sigma}$. This representation allows us to vary the shape of the distribution and the variance independently, to study different measures of risk.

The sender privately learns the realization of the state θ. The receiver can choose to communicate with the sender (*communication*). In this case, a sender strategy maps states into distributions over messages, $M_S : \mathcal{S} \to \Delta M$; and a receiver strategy maps messages into actions $Y_R : M \to \mathbb{R}$. Strict concavity of payoffs implies that a restriction to pure receiver strategies is without loss of generality. As a simple alternative, the receiver can choose to delegate decision-making to the sender (*delegation*) in which case a sender strategy maps states into actions, $Y_S : \mathcal{S} \to \mathbb{R}$. We solve for Bayes Nash equilibria of the game.

3 Equilibria and Payoffs

3.1 Equilibria of the communication game

As is standard in cheap talk games satisfying the single crossing condition, equilibria are partitional. A partitional equilibrium is characterized by a sequence of *critical types*, $\boldsymbol{t}^n = (t_i^n)$, with $t_{i-1}^n < t_i^n$ and n relating to the number of induced actions. Sender types strictly within an interval, $\left(t_{i-1}^n, t_i^n\right)$, induce the same action; critical types, t_i^n, are indifferent between inducing the action in the interval below or the action in the interval above. As we show in Proposition 1 below, for any finite number of induced actions equilibria are symmetric in our model. For notational simplicity we, therefore, take $t_i^n \geq 0$ and denote the critical types below zero by $-t_i^n$ for all i and

[7] Note that any symmetric one-dimensional density is elliptical (Cambanis et al. (1981)). The particular representation of elliptical densities can be found, e.g., in Gómez et al. (2003). Many distributions that are used in economics are elliptical with logconcave densities. Examples include the uniform, the Gaussian, the Laplace, and many more.

n. Receiving a message that indicates that $\theta \in [\underline{t}, \bar{t})$, the receiver updates her belief by forming the conditional expectation $\mu(\underline{t}, \bar{t}) = \mathbb{E}\left[\Theta | \, \Theta \in [\underline{t}, \bar{t})\right]$. For equilibrium critical types $\boldsymbol{t}^n$, we define

$$\mu_i^n := \mathbb{E}\left[\Theta | \, \Theta \in \left[t_{i-1}^n, t_i^n\right)\right] \quad \text{for } i = 1, \ldots, n \text{ and } \quad \mu_{n+1}^n := \mathbb{E}\left[\Theta | \, \Theta \geq t_n^n\right]. \tag{1}$$

Thus, the receiver's equilibrium action given a message indicating $\theta \in \left[t_{i-1}^n, t_i^n\right)$ is $a \cdot \mu_i^n = \arg\max_y \mathbb{E}[u_R(y, \theta, a) | \theta \in \left[t_{i-1}^n, t_i^n\right)]$. The indifference conditions of critical types that determine partitional equilibria are given by

$$t_i^n - a \cdot \mu_i^n = a \cdot \mu_{i+1}^n - t_i^n, \qquad \text{for } i = 1, \ldots, n. \tag{2}$$

Symmetric equilibria come in two classes, depending on whether the total number of induced actions is even or odd. In an equilibrium with an even number of actions, type $\theta = 0$ must be a critical type. We call this type of equilibrium an *Even equilibrium*, and the characterization uses $t_0^n = 0$. If the total number of induced actions is odd, then a symmetric interval around zero is part of the equilibrium. We call this an *Odd equilibrium*. In this case, we omit t_0^n from the construction. For an illustration with $n = 1$, see Figure 1. The step function depicts the receiver's actions.

Proposition 1 *Assume a symmetric distribution with a logconcave density.*
i) For all n, there exist an essentially unique Even equilibrium, that is symmetric and induces $2\,(n+1)$ *actions, and an essentially unique Odd equilibrium, that is symmetric and induces* $2n + 1$ *actions.*
ii) Even and Odd equilibrium thresholds and actions converge for $n \to \infty$. *The limits define equilibria that induce infinitely many actions; we call them limit equilibrium.*[8] *In particular, we have* $\lim_{n\to\infty} t_1^n = 0$ *and* $\lim_{n\to\infty} t_n^n < \infty$.

Part i) of Proposition 1 proves the existence and uniqueness of partitional equilibria for arbitrary finite n. An analogous characterization of partitional equilibria is

[8]We do not rule out the existence of other infinite equilibria, we focus on limit equilibria throughout the paper.

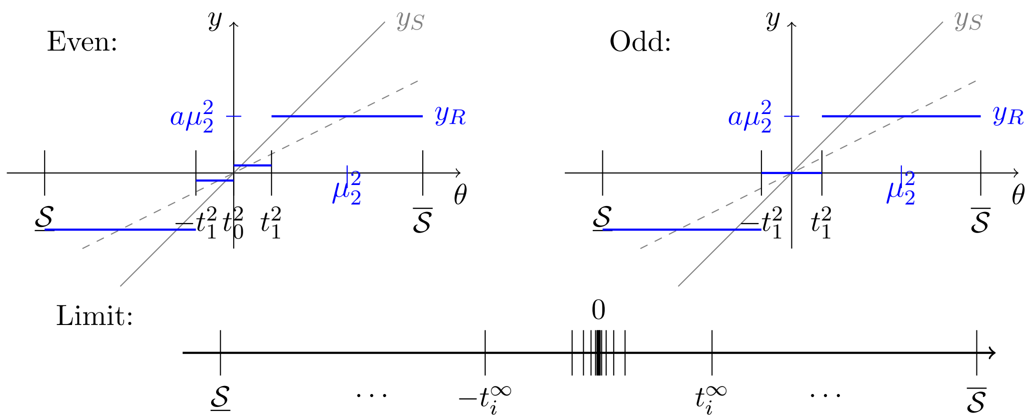

Figure 1: Partitional equilibria. Even and Odd equilibria for $n = 1$. In a limit equilibrium, intervals around the prior mean 0 get arbitrarily small as $n \to \infty$.

given in Deimen and Szalay (2019) for the special case of the Laplace distribution. Proposition 1 generalizes the result to all symmetric distributions with a logconcave density. Note that the support can be bounded or unbounded. Logconcavity of the distribution and alignment $a \in (0, 1)$ together imply that the solution of a certain forward difference equation is monotonic in the initial value, which we use to prove uniqueness.[9]

Part ii) of the proposition proves that the limit as $n \to \infty$ also is an equilibrium. A limit equilibrium features an accumulation point at zero and a finite highest critical type, $\lim_{n\to\infty} t_n^n < \infty$, even if the support is unbounded. The reason is that for a distribution with a logconcave density the mean residual life, $\mathbb{E}[\theta - t_n^n | \theta > t_n^n]$, is decreasing towards zero as $t_n^n \to \infty$. Equilibrium condition (2) implies a lower bound on the distance between t_n^n and the induced action below, μ_n^n. Since this distance in equilibrium must equal the distance between t_n^n and the induced action above, μ_{n+1}^n, we get a lower bound for the mean residual life. Therefore, the highest critical type must be finite to ensure that the mean residual life is sufficiently large. The insight

[9] For the proof, we take equilibria as a combination of a "forward solution" and a "closure condition."A forward solution that starts at $t_0 = 0$, takes the length of the first interval, say $t_1 = \tau$, as given, and computes the "next" threshold, $t_2(\tau)$, as a function of the preceding two, τ and t_0. Likewise, all following thresholds are constructed using their two predecessors. The closure condition for an equilibrium with n positive thresholds requires that τ is such that type $t_n^n(\tau)$ satisfies the indifference condition.

that there is a finite highest critical type even if the support is unbounded is new to the literature, which typically assumes a compact state space.

The partition of a limit equilibrium is illustrated in Figure 1, bottom panel. While the partitional form of equilibria is known from the seminal work of Crawford and Sobel (1982), the structure of the limit equilibrium is closest in spirit to Alonso et al. (2008) and Rantakari (2008). Gordon (2010) offers the first systematic account of the existence of infinite equilibria. We add to this literature by highlighting the role of distributions and, in particular, the role of logconcavity for existence and uniqueness. Logconcavity provides a microfoundation for regularity properties that are often imposed in the literature.[10]

3.2 Communication gains

Define the random variable μ^n of conditional expectations on the discrete support $(\pm\mu_i^n)_{i=1}^{n+1}$, with μ_i^n (given in equation (1)) derived from the equilibrium partition (t_i^n). As is standard in cheap talk games with quadratic losses, the expected equilibrium utility is a function of the *expected residual variance* after communication, $\mathbb{E}[(\sigma^2)^n]$, where $(\sigma^2)^n$ is the random variable of conditional variances conditional on the equilibrium partition. The expected residual variance measures the expected uncertainty left after communication has taken place. By the *law of total variance*, the expected residual variance equals the prior variance minus the variance of the inferred posterior means after communication,

$$\mathbb{E}[(\sigma^2)^n] = \sigma^2 - var\left(\mu^n\right).$$

The variance of the inferred posterior means $var\left(\mu^n\right)$ measures the expected informational *gain from communication*. Communication performs better if the expected residual variance is smaller, or equivalently, if the variance of the inferred posterior

[10]Following Crawford and Sobel (1982), the literature invokes condition M to ensure uniqueness. Logconcavity of the density and a receiver response with a slope less than one – not necessarily constant – is a condition on the primitives of the model that ensures that condition M is satisfied. Gordon (2010) assumes a *regular* receiver response. Our results are in line with his insightful characterization. We provide conditions that make a receiver response regular.

means is higher. For our comparative statics analysis, it turns out that the latter object is analytically more convenient to work with.

3.3 Scale: linear transformations and equilibrium payoffs

As the next two lemmas show: our model is *scalable*. This means that equilibrium strategies are linear in the standard deviation and that utilities are linear in the variance.

Lemma 1 *Fix the shape of the distribution $\psi(\cdot)$.*
i) Equilibrium strategies (t_i^n) and $a \cdot (\mu_i^n)$ are linear in the standard deviation σ:
$\boldsymbol{z}^n = (z_i^n) = \left(\frac{t_i^n}{\sigma}\right)$ *is the sequence of equilibrium critical types for the standardized distribution with unit variance, and* $\mathbb{E}\left[\Theta\right|\Theta \in \left[t_{i-1}^n, t_i^n\right]\right] = \sigma\mathbb{E}\left[Z\right| Z \in \left[z_{i-1}^n, z_i^n\right]\right]$*,* *for* $Z := \frac{\Theta}{\sigma}$*.*
ii) The receiver's expected utility in any equilibrium of the communication game is a linear function of the variance σ^2:

$$\mathbb{E}u_R^{com}(y_R, \Theta, a) = -a^2\left(\sigma^2 - var(\mu^n)\right) = -a^2\left(1 - \ell(a,n)\right)\sigma^2,$$

for some function $\ell(a,n)$ that is independent of σ^2.

The first statement follows from a change of variables to the standardized random variable $Z = \frac{\Theta}{\sigma}$, which shows the equivalence of the original indifference conditions to the ones with standardized critical types. The second statement uses the law of total variance to write the receiver's equilibrium expected utility as a function of $\sigma^2 - var(\mu^n)$ in place of the expected residual variance (as explained above). The proof shows that the gain from communication $var(\mu^n)$ is linear in the variance. This follows since the probability distribution over the equilibrium receiver actions is not affected by the standard deviation and the conditional means are linear in the standard deviation by part i).

As a simple alternative to communicating with the sender, we consider simple, unconstrained delegation to the sender.[11] Under delegation, there is no loss of information as the informed sender takes the action $y_S = \theta$. Sender and receiver, however, disagree on the optimal action by $(1 - a)$. This immediately implies the following lemma.[12]

Lemma 2 *The receiver's expected utility under delegation is*

$$\mathbb{E}u_R^{del}(y_S, \Theta, a) = -(1-a)^2 \sigma^2.$$

Hence, expected utilities, whether arising from communication or from delegation, are linearly decreasing in the variance. A higher variance, thus, results in lower expected utilities under both institutions. A higher variance, however, by linearity, never results in a change of the optimal choice of institution in our model.

Corollary 1 *Fix the shape of the distribution $\psi(\cdot)$. The choice between delegation and communication – in any equilibrium of the communication game – is independent of the variance σ^2.*

This is a direct consequence of σ being a scale variable.[13] By implication, if one mode of decision-making is better than the other for some level of variance then it is better for any level of variance. In other words, when comparing the optimal choice of institution for two different distributions it is without loss of generality to focus on distributions with the *same variance*:

Observation 1 *Consider two distributions with shapes $\psi_1(\cdot)$ and $\psi_2(\cdot)$. The optimal choice of institution is the same for both distributions if and only if it is the same for both distributions where one of them is rescaled so that both have equal variances.*

[11] By contrast, under optimal delegation, the receiver can constrain the choice set of the sender. See, for example, Alonso and Matouschek (2008). Optimal delegation can replicate communication outcomes and is therefore always weakly better. We show that even simple, unconstrained delegation can strictly improve upon communication.

[12] Note that communication dominates delegation for $a \leq \frac{1}{2}$. The reason is that even a babbling equilibrium, which is the worst among all equilibria of the communication game, results in a payoff of $-a^2\sigma^2$.

[13] Equivalent observations have been made in the literature in models in which the state follows a uniform distribution. We extend the result to all scalable distributions. The shape of the distribution does not matter, as long as it is kept fixed.

Thus, if the aim is to identify factors that affect the choice of institution, we can focus on distributions with the same variance. This motivates our focus on more complicated comparisons of distributions.

4 Shape: convex transformations

By Observation 1, differences in variances across distributions are not the driving force behind changes in the choice of institution. For this reason, we net out differences in variances in what follows. Instead of considering risk in terms of linear transformations (scale), we now focus on risk in terms of convex transformations (shape). We study the implications of spreads in the distributions that leave mean and variance constant, but do increase higher (even) moments.[14] In particular, we combine pairs of different stochastic orders which all have the feature that they provide comparisons of the shape of the distributions with respect to how much mass and how much variability lies in the tails. We consider stochastic orders that describe higher risk and that make a difference in the choice of the optimal institution.

4.1 Convex transformations and equilibrium quantiles

Consider two distinct random variables Θ_f and Θ_g with distributions F and G and densities f and g, respectively. Let $\Theta_{f+} := |\Theta_f|$ and $\Theta_{g+} := |\Theta_g|$ denote the absolute values of these random variables (or equivalenty, by symmetry, the random variables with distributions truncated to the positive halves of their supports) with densities f_+ and g_+ and cdfs F_+ and G_+, respectively. By symmetry, it is without loss of generality and analytically convenient to study the (one-sided) half-distributions F_+ and G_+. The economic intuition, however, is easier to convey by means of the two-sided distributions F and G. We, therefore, go back and forth between the two representations in what follows.

We assume throughout the paper that random variable Θ_g is more variable than

[14] We emphasize that the assumption of a constant variance rules out that the halves are stochastically ordered (FOSD) and the distributions overall are higher in the convex order (SOSD).

random variable Θ_f in the sense of *nested supports*, $\mathcal{S}_f \subseteq \mathcal{S}_g$. For bounded supports, we assume $\mathcal{S}_f \subset \mathcal{S}_g$.

Moreover, we assume throughout the paper that Θ_{g+} results from a convex transformation of Θ_{f_+} in the following sense:

Definition 1 *(van Zwet (1964))* Θ_{f_+} *is smaller than* Θ_{g+} *in the convex transform order,* $\Theta_{f_+} \leq_c \Theta_{g+}$*, if*

$$G_+^{-1} F_+ (\theta) \ \textit{ is convex in } \theta \textit{ on the support of } F_+. \tag{CTO+}$$

Intuitively, distribution G_+ is more skewed than F_+ towards high realizations on the support. Because of the common origin at $\theta = 0$, convexity of $G_+^{-1} F_+ (\cdot)$, and $\overline{\mathcal{S}}_f \leq \overline{\mathcal{S}}_g$, there exists a $\hat{\theta}$ such that $G_+^{-1} F_+ (\hat{\theta}) = \hat{\theta}$. This implies that $1 - G_+ (\theta) < 1 - F_+ (\theta)$ for $\theta \in (0, \hat{\theta})$ and $1 - G_+ (\theta) > 1 - F_+ (\theta)$ for $\theta \in (\hat{\theta}, \overline{\mathcal{S}}_g)$. Thus F_+ is stochastically higher than G_+ below $\hat{\theta}$ and vice versa above $\hat{\theta}$. By symmetry, G has more mass in the tails than distribution F. Formally, if Θ_{f+} and Θ_{g+} satisfy CTO+, then Θ_f is smaller than Θ_g in van Zwet's S-order, $\Theta_f \leq_S \Theta_g$ (van Zwet (1964)), which implies a higher kurtosis.

The convex transform order orders distributions independently of location and scale (van Zwet (1964)). Therefore, rescaling distributions with bounded supports to a common support preserves the ranking of the distributions according to CTO+. The intersection point $\hat{\theta}$, however, depends on scale.

Proposition 2 *Consider two distributions F, G such that CTO+ holds. Then the quantiles at the equilibrium thresholds under the respective distributions, $t_{i,f}^n, t_{i,g}^n$, satisfy $F_+ \left(t_{i,f}^n \right) \leq G_+ \left(t_{i,g}^n \right)$ for all n.*

The proof of the proposition uses convexity and Jensen's inequality. It shows that under CTO+ the quantiles at the equilibrium thresholds are ordered. That is, the quantile at the i^{th} threshold under G_+ is higher than the quantile at the i^{th} threshold under F_+. Note that the ordering only refers to the quantiles at the thresholds but not to the thresholds themselves. The quantiles at the thresholds determine the equilibrium probability distribution over the receiver's actions. Hence, the receiver is more likely to take relatively higher indexed actions under F_+ than under G_+. For an illustration, see Figure 2 left panel.

4.2 The effects of shape and scale on equilibrium strategies

In this subsection, we assume that the supports are bounded. We first scale the random variables to a common support, which is useful for a meaningful comparison of equilibrium strategies. We then consider a stochastic order of distributions on the common support.[15] We provide comparative statics of equilibria and show that standard stochastic comparisons cannot be used to compare payoffs.

Consider the scaled random variable $\Theta_{\hat{f}_+} = \frac{\overline{\mathcal{S}}_g}{\overline{\mathcal{S}}_f} \cdot \Theta_{f_+}$, with density $\hat{f}_+$ and cdf $\hat{F}_+$ on support $\theta \in \left[0, \overline{\mathcal{S}}_g\right]$.

Definition 2 *The distributions $\hat{f}_+$ and g_+ – scaled to the same support – satisfy the monotone likelihood ratio property, if*

$$\frac{\hat{f}_+(\theta)}{g_+(\theta)} \text{ is increasing in } \theta. \tag{MLRP$\hat{+}$}$$

Condition MLRP$\hat{+}$ implies that the distributions are ordered in the standard stochastic order (FOSD) on the positive half of the support, thus $\hat{\theta} = \overline{\mathcal{S}}_g$. It says that, scaled to the same support, higher realizations of θ are relatively more likely under distribution $\hat{F}_+$ than under distribution G_+. On the entire support, the likelihood ratio $\frac{\hat{f}(\theta)}{g(\theta)}$ is U-shaped so that more extreme realizations are relatively more likely under $\hat{F}_+$ than under G_+. This has the following impact on equilibrium choices.

Proposition 3 *Consider two distributions F, G such that MLRP$\hat{+}$ holds. Then the equilibrium critical types and induced actions satisfy $\frac{t^n_{i,f}}{\overline{\mathcal{S}}_f} > \frac{t^n_{i,g}}{\overline{\mathcal{S}}_g}$ and $\frac{\mu^n_{i,f}}{\overline{\mathcal{S}}_f} > \frac{\mu^n_{i,g}}{\overline{\mathcal{S}}_g}$ for all $i \leq n$.*

[15] For unbounded supports, considered in the next subsection, we will introduce another order consistent with the one provided here.

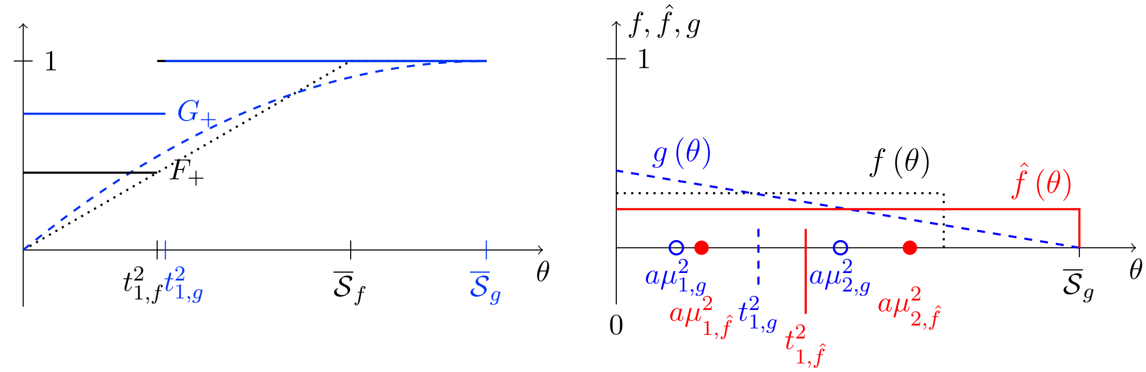

Figure 2: Proposition 2 left panel and Proposition 3 right panel, for uniform distribution F (black dotted) and triangular distribution G (blue dashed).

If the distributions satisfy MLRP$\hat{+}$, then the i^{th} threshold type and the i^{th} induced action by the receiver are farther away from the prior mean under $\hat{F}_+$ than under G_+. Since equilibrium actions are linear in scale, we can scale $\hat{F}_+$ back to the original support to obtain the comparison relative to the lengths of the supports of F_+ and G_+.[16] For an illustration, see Figure 2 right panel.

Corollary 2 *Suppose F, G satisfy CTO+ and MLRP$\hat{+}$, then the distribution of $\frac{\mu^n_f}{\overline{S}_f}$ is a mean-preserving spread of the distribution of $\frac{\mu^n_g}{\overline{S}_g}$, implying $var_f\left(\frac{\mu^n_f}{\overline{S}_f}\right) > var_g\left(\frac{\mu^n_g}{\overline{S}_g}\right)$.*

If both conditions CTO+ and MLRP$\hat{+}$ hold together, then the distribution of receiver equilibrium actions under F corrected for the length of the support is a mean-preserving spread of the distribution of receiver equilibrium actions under G. As is well known, if one distribution is a mean-preserving spread of another distribution, then the former has a higher variance. The reverse is not true. Hence, to conclude that $var_f\left(\frac{\mu^n_f}{\overline{S}_f}\right) > var_g\left(\frac{\mu^n_g}{\overline{S}_g}\right)$ it is sufficient but not necessary that the distribution of $\frac{\mu^n_f}{\overline{S}_f}$ is a mean-preserving spread of the distribution of $\frac{\mu^n_g}{\overline{S}_g}$.

Propositions 2 and 3 deliver clear predictions about the comparative statics of equilibria. For the 'less risky' distribution F, (i) the probability of the receiver taking higher indexed actions is higher (Proposition 2) and (ii) the scaled equilibrium

[16]That MLRP orders equilibrium actions is known from Chen and Gordon (2015). We add the aspect of scale.

threshold types and induced actions are farther away from the prior mean (Proposition 3). The scale correction inherent in MLRP$\hat{+}$, however, is not very useful for the comparative statics of the expected utilities. The reason is that the ordering of the gains from communication $var_f\left(\mu_f^n\right) > \left(\frac{\overline{S}_f}{\overline{S}_g}\right)^2 var_g\left(\mu_g^n\right)$ may result from $\frac{\overline{S}_f}{\overline{S}_g} < 1$ being sufficiently small, instead of from the desired conclusion that $var_f\left(\mu_f^n\right) > var_g\left(\mu_g^n\right)$.

4.3 The effect of shape on equilibrium gains

Our next results allow us to compare distributions with respect to their gain from communication – irrespective of supports. We drop condition MLRP$\hat{+}$ and impose instead a ranking of distributions that also holds for unbounded supports. We assume that random variable Θ_{g+} is *uniformly more variable* than random variable Θ_{f+} in the following sense.

Definition 3 *(Whitt (1985)) Θ_{f+} is smaller than Θ_{g+} in the uniform conditional variability order, $\Theta_{f+} \leq_{uv} \Theta_{g+}$, if the support of Θ_{f+} is a subset of the support of Θ_{g+}, $\mathcal{S}_{f+} \subseteq \mathcal{S}_{g+}$, and the ratio*

$$\frac{f_+(\theta)}{g_+(\theta)} \text{ is unimodal over } \mathcal{S}_{g+}\text{, where the mode is a supremum,} \tag{UCV+}$$

but Θ_{f+} and Θ_{g+} are not ordered by the standard stochastic order.

Figure 3 illustrates. The top panel depicts two densities f, g where g features a relatively higher likelihood of extreme outcomes than f. The bottom panel depicts the likelihood ratio, $\frac{f(\theta)}{g(\theta)}$. On the positive half of the support, $\frac{f_+(\theta)}{g_+(\theta)}$ is unimodal with interior mode m.

As we show in the proof of Proposition 4 below, $\Theta_{f+} \leq_{uv} \Theta_{g+}$ implies (again) that there exists some $\hat{\theta}$ such that $1 - G_+(\theta) < 1 - F_+(\theta)$ for $\theta \in (0, \hat{\theta})$ and $1 - G_+(\theta) > 1 - F_+(\theta)$ for $\theta \in (\hat{\theta}, \overline{\mathcal{S}}_g)$. Thus, UCV+ implies, consistently with CTO+ that the distribution of Θ_g has more mass in the tails than the distribution of Θ_f.

Proposition 4 *Suppose that the densities f and g are logconcave and induce the same variance σ^2. Let $\frac{f_+}{g_+}$ satisfy UCV+ and let F_+ and G_+ satisfy CTO+. Then,*

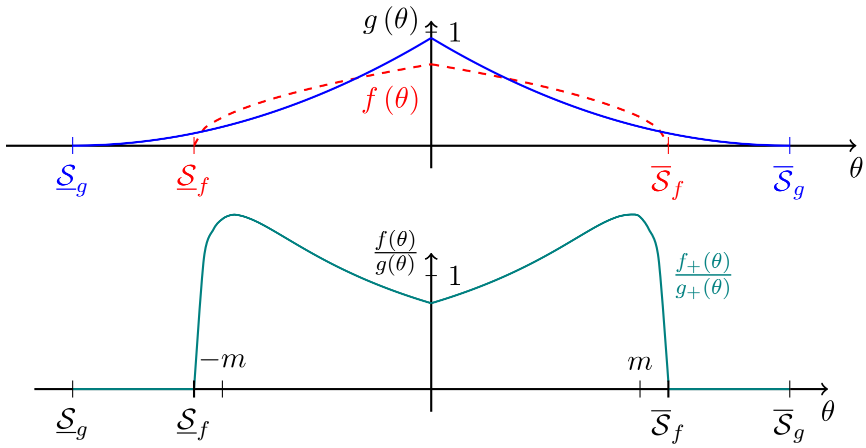

Figure 3: Top: distributions f, g satisfying UCV+. Bottom: the likelihood ratio $\frac{f}{g}$.

there exists $a' \in (0,1)$, defined in the appendix, such that for $a \leq a'$, the distribution of μ_f^n is a strict mean preserving spread of the distribution of μ_g^n, implying that

$$var_f(\mu_f^n) > var_g(\mu_g^n).$$

Condition CTO+ implies that for any alignment $a \in (0,1)$ the equilibrium probability distribution over the receiver's actions for density f puts more weight on the more extreme actions than g. For a relatively low, all thresholds and receiver actions are relatively close to the prior mean. Due to UCV+ this implies that the receiver's equilibrium actions are all farther away from zero under distribution f than under distribution g. The two effects – the ranking of quantiles and actions – taken together imply that the distribution of μ_f^n is a mean preserving spread of the distribution of μ_g^n.

Note that our assumption in Proposition 4 on the level of a is sufficient but not necessary. In particular, $a \leq a'$ is sufficient, not necessary to obtain a mean preserving spread, and a mean preserving spread is sufficient, not necessary to obtain a ranking of variances. Increasing a above a' will eventually result in distributions that are no longer comparable in terms of mean preserving spreads. Our result, however, only requires the weaker condition of a ranking of the variances. Since the gains from communication are continuous in a, the ranking of variances is preserved for a larger set of alignment parameters beyond a':

Corollary 3 *Suppose that the densities f and g are logconcave and induce the same variance σ^2. Let $\frac{f_+}{g_+}$ satisfy UCV+ and let F_+ and G_+ satisfy CTO+. Then, there exists $a'' > a'$ such that for $a \leq a''$, $var_f(\mu_f^n) > var_g(\mu_g^n)$.*

Under our assumptions, there is more information transmission and the expected utilities are higher under f than under g. For both distributions, a higher marginal cost of adaptation pushes all choices closer to the prior mean; there is less adaptation. However, the less risky distribution is more resistant against this force, because its half-distribution is stochastically higher close to zero. We conclude that a higher weight in the tails is detrimental to the quality of information transmission, provided that interests are 'sufficiently misaligned.'

5 Delegation versus communication

In this section, we quantify the effects derived in the previous analysis. We provide some meaning to the term 'sufficiently misaligned' interests used in Proposition 4 and Corollary 3. Moreover, we link our findings back to the comparison of delegation versus communication. We show that for a given alignment of preferences, a more risky distribution can change the optimal way of decision-making from communication to delegation. All distributions we compare in this section are ranked according to CTO+ and UCV+.

To quantify the effects and to derive a formula for the gain from communication, we rely on a 'dynamic programming' procedure as our technical tool. The slope of the *tail-truncated expectation* function $\phi(t) := \mathbb{E}[\Theta | \Theta \geq t]$ for $t \geq 0$, is a crucial determinant of this value. The case of a linear tail-truncated expectation, which is satisfied by the two-sided generalized Pareto distribution is particularly structured. We treat this in the next subsection. A second class of interest is the class of convex tail-truncated expectation functions. The Gauss distribution is a prominent case with this property, which we treat thereafter.

The gain from communication can be quantified as follows.

Proposition 5 *Suppose that $\phi(t) := \mathbb{E}[\Theta | \Theta \geq t]$ is convex in $t \geq 0$. Then the variance of μ^n in a limit equilibrium satisfies*

$$var(\mu^\infty) \geq \frac{2}{2 - a \cdot \phi'(0)} \cdot \phi(0)^2.$$

If $\phi(t)$ is linear in $t \geq 0$, then the condition is satisfied with equality.

The (lower bound on) the variance of μ^n in a limit equilibrium is a product of two terms. The factor $\phi(0)^2 = \mathbb{E}[\Theta | \Theta \geq 0]^2$ measures the amount of information that is transmitted by binary communication, when dividing the state space into positive and negative realizations. The factor $\frac{2}{2-a\cdot\phi'(0)}$ captures (a lower bound on) the additional information contained by dividing each half into a countable infinity of subintervals. The latter term depends on the slope of the tail-truncated expectation, $\phi'(t)$. The slope is constant for a linear tail-truncated expectation function which therefore yields a closed form solution. The slope is increasing for convex tail-truncated expectations. We thus obtain a lower bound on the variance of equilibrium actions by using the minimal slope of the conditional expectation – which amounts to the slope at zero, $\phi'(0)$.

The computation is based on a procedure which is akin to dynamic programming (for $a = 1$, i.e., identical sender and receiver preferences, it would be dynamic programming in the literal sense). In particular, we compute the expected squared deviation from $\phi(0)$ conditional on the last interval, then conditional on the last two intervals, and so on, proceeding towards zero. In each step, we can simplify the expression using the indifference condition of the threshold types, the law of iterated expectations – which links expectations over subintervals to expectations truncated to the tail of the distribution – and the special form of tail-truncated expectations. If the tail-truncated expectation function is linear in the truncation point, we can carry an exact functional form backwards towards zero and in the limit obtain a closed form expression. The procedure was developed in (Deimen and Szalay (2019)). Here, we generalize the procedure to the case of convex tail-truncated expectations. For this case, we show that at each step of the procedure, we obtain a lower bound on the expected squared deviation from $\phi(0)$. Thus in the limit, we derive a lower bound on the expected gain from communication in a limit equilibrium.

Note that our quantitative assessment of communication gains via 'dynamic programming' applies to any distribution that becomes relatively more variable towards the tail of its distribution in the sense of a globally increasing residual coefficient of variation (Gupta and Kirmani (2000)).[17]

5.1 The linear case: generalized Pareto distribution

The tail-truncated expectation $\phi(t) = \mathbb{E}[\Theta | \Theta \geq t]$ is linear in $t \geq 0$ if the state is distributed according to a *two-sided generalized Pareto distribution.* For this class, the density is

$$f(\theta; \delta, s) = \frac{1}{2s}\left(1 + \delta\frac{|\theta|}{s}\right)^{-\frac{1}{\delta}-1} \text{ for } \theta \in \left[\frac{s}{\delta}, -\frac{s}{\delta}\right], \tag{GP}$$

where $s \in (0, \infty)$ is a scale parameter and $\delta \in [-1, 0]$ is a shape parameter.[18] The variance of the distribution is $\sigma^2(s, \delta) = \frac{2s^2}{(1-\delta)(1-2\delta)}$.

The shape parameter δ and the scale parameter s can be changed independently. Increases in scale s make the support, $[\frac{s}{\delta}, -\frac{s}{\delta}]$, wider and move equilibrium actions further way from the mean. Increases in shape δ move equilibrium actions closer to the mean, when controlling for the support. It can be shown that the distributions in this generalized Pareto class satisfy the definitions CTO+, MLRP$\hat{+}$, and UCV+.

The class is rich as it nests many well-known distributions. In particular, the case $\delta = -1$ is the uniform distribution, $\delta = -\frac{1}{2}$ is the triangular distribution, and the limit case $\delta = 0$ is the Laplace distribution. For an illustration of these examples, see

[17] Gupta and Kirmani (2000) show that the residual coefficient of variation, i.e., the ratio of residual variance and mean residual life squared, increases in the truncation point if $\phi(t)$ is convex in t.

[18] The distribution is constructed from the well-known one-sided generalized Pareto by reflecting at zero. The location parameter is set to zero, to ensure that the mean is zero. The distribution is defined more generally for shape parameters $\delta \in (-\infty, \infty)$, but we restrict attention to the subset that features logconcave tails. We treat the case $\delta \geq 0$ in Deimen and Szalay (2019); these distributions have logconvex tails and an infinite support.

Figure 4.

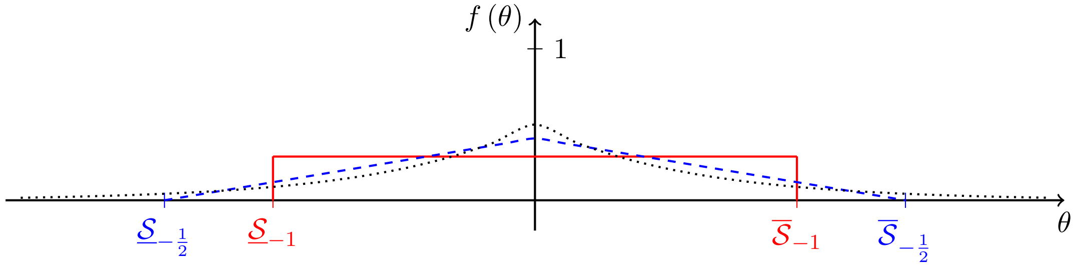

Figure 4: The uniform distribution (solid red, $\delta = -1$) and the triangular distributions (dashed blue, $\delta = -\frac{1}{2}$) and the Laplace distribution (dotted black, $\delta = 0$) all with variance $\sigma^2 = 1$.

By Proposition 5, for the generalized Pareto environment the expected utilities arising from communication can be stated in closed form.[19]

Lemma 3 *For the two-sided generalized Pareto distribution with shape $\delta \in [-1, 0]$ and scale $s^2 = \sigma^2 \frac{(1-\delta)(1-2\delta)}{2}$, we have that $\phi'(0) = \frac{1}{1-\delta}$. Hence, in a limit equilibrium, we have*

$$var(\mu^\infty) = \frac{2}{2 - \frac{a}{1-\delta}} \phi(0)^2 = \frac{2 - \frac{1}{1-\delta}}{2 - \frac{a}{1-\delta}} \sigma^2. \tag{3}$$

The second equality in (3) obtains from noting that $\phi(0) = \frac{s}{1-\delta}$ and using the functional form of the variance. Naturally, $var(\mu^n) \leq var(\mu^\infty) \leq var(\Theta)$. For $a \to 0$, the value $var(\mu^\infty)$ approaches the value of binary communication. Note that for given alignment $a \in (0, 1)$, the value $var(\mu^\infty)$ is decreasing in δ.[20] A larger shape parameter reduces the value of communication, less information is transmitted in equilibrium. Thus, within the generalized Pareto class, the shape parameter has a strictly negative impact on the gain from communication for any value of $a < 1$. In the sense of Corollary 3, for this class, the condition of 'sufficiently misaligned' interests is always satisfied, i.e., $a'' = 1$, and there is no restriction.

[19] In Deimen and Szalay (2019), distributions with a linear tail-truncated expectation are derived from first principles as the solution to a differential equation. In that formulation, we obtain a solution that involves variance and the slope of the tail-truncated expectation. Here, we observe that the generalized Pareto class can be obtained as a re-parametrization – in terms of shape and scale – of the distributions with linear tail-truncated expectations.

[20] For $a = 1$, the value of partitional communication reaches the upper bound of fully revealing communication. For $a = 0$, the receiver's action equals zero for any sender strategy.

It is now straightforward to investigate the effect of the shape of the distribution on the optimal choice of institution – communication versus delegation.

Proposition 6 *Suppose the receiver can choose between communication and delegation. Then, delegation is better than communication – in any equilibrium of the communication game – if* $\delta \geq \frac{2-3a}{2-2a}$. *Communication in a limit equilibrium is better than delegation if* $\delta \leq \frac{2-3a}{2-2a}$.

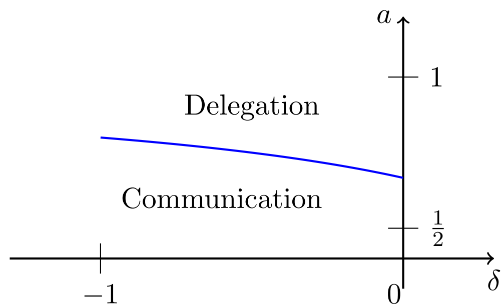

Figure 5: Delegation versus communication. On the horizontal axis, the shape parameter δ increases from -1 (uniform distribution) to 0 (Laplace distribution); on the vertical axis, the level of alignment a increases from $\frac{1}{2}$ to 1.

While the performance of delegation depends only on the variance of the environment, the performance of communication depends in addition on the shape of the distribution. The fraction of information that is transmitted in a limit equilibrium, $\frac{2-\frac{1}{1-\delta}}{2-\frac{a}{1-\delta}}$, is smaller in environments that feature more shape, i.e., larger δ. We depict the comparison in Figure 5.

Consistent with the literature, delegation dominates communication when the interests are relatively well aligned and the receiver is quite responsive to the sender's advice, $a \geq \frac{2-2\delta}{3-2\delta}$.[21] The comparison in terms of shape adds a new dimension to the literature. For $a \in \left(\frac{2}{3}, \frac{4}{5}\right)$, for a distribution with a low shape parameter communication is optimal but for a distribution with a higher shape parameter delegation is optimal. In other words, an increase of the mass in the tail of the distribution induces a change in the mode of decision-making from communication to delegation in the named range.

[21]See, for example, Alonso et al. (2008) and Rantakari (2008) who study a uniform distribution, i.e., $\delta = -1$. See also Dessein (2002).

5.2 A convex case: Gauss versus Laplace

A leading example within the class of distributions with convex tail-truncated expectations is the Gauss distribution (see Sampford (1953)).

Lemma 4 *For the Gauss distribution, $\phi'(0) = \frac{2}{\pi}$. The gain from communication in a limit equilibrium is bounded from below,*

$$var(\mu^{\infty}) \geq \frac{2}{2 - a\frac{2}{\pi}} \phi(0)^2 = \frac{2}{\pi - a} \sigma^2. \tag{4}$$

By Proposition 5, for the Gauss distribution the expected utilities arising from communication can be bounded from below. The lower bound on the variance of equilibrium actions uses the minimal slope of the truncated expectation – which amounts to the slope at the origin, $\phi'(0) = \frac{2}{\pi}$. The equality in (4) results from the fact that for the Gauss distribution $\phi(0)^2 = \frac{2}{\pi}\sigma^2$.

With this at hand, we can compare delegation to communication under the Gauss distribution and under the Laplace distribution. The Laplace distribution has the largest shape parameter $\delta = 0$ in the class of logconcave generalized Pareto distributions. In this sense, this is the most risky distribution in this class.

Note that the Gauss and the Laplace distributions are ordered in the convex transform order and that they satisfy the uniform conditional variability order (see Lemma A.8 in the appendix). Interests are sufficiently misaligned in the sense of Corollary 3 if $a \leq a'' = 0.858$. The lower bound on the value of communication under the Gauss distribution outperforms the exact value of communication under the Laplace distribution. Using these values, we identify situations in which communication is optimal for the Gauss distribution, while delegation is optimal for the Laplace distribution.

Proposition 7 *If communication is preferred over delegation for the Laplace distribution, then communication is also preferred over delegation for the Gauss distribution. Conversely, there is a nonempty set of preference alignment parameters a for which delegation is preferred for the Laplace distribution whereas communication is preferred for the Gauss distribution.*

To rephrase the proposition, in the particular case at hand, there is (again) more delegation compared to communication when the environment features more mass in the tails in the sense of CTO+ and UCV+.

To understand the proposition more formally, recall that the locus of indifference points between communication and delegation for the Laplace distribution can be computed in closed form. In particular, communication is optimal for the Laplace for any $a \leq \frac{2}{3}$. Recalling that the lower bound on the value of communication under the Gauss distribution outperforms the value from communication under the Laplace distribution for $a \leq 0.858$ implies the first statement. To prove the converse statement, we show that the lower bound on the value of communication under the Gauss distribution trumps delegation for any $a \leq 0.702$. Since already for $a > \frac{2}{3}$, delegation is preferred for the Laplace, there is a set of preference alignment parameter values, $a \in (0.667, 0.702)$, for which communication performs better for a Gauss distribution, while delegation is better for the more risky Laplace distribution.

The Gauss distribution is one example. Similar results can be obtained for any distribution with a logconcave density and a convex tail-truncated expectation function.

6 Thin versus heavy tails

We have so far confined our attention to distributions with a logconcave density. These feature relatively thin tails – logconcave densities have thinner tails than the Laplace distribution, which features loglinear tails. We have used this restriction in Proposition 1 to prove uniqueness and existence. We now drop the assumption to discuss distributions with *heavy tails* – tails that are heavier than those of the Laplace distribution. Note that we keep the assumption that the variance is finite, and thus still impose a restriction that the tails cannot be too heavy.

We note that dropping logconcavity does not imply that equilibria necessarily cease to exist, nor that there are necessarily multiple equilibria inducing a given number of receiver actions. It may still be the case that *Condition M* holds (Crawford and Sobel (1982)) or that the receiver responses are *regular* (Gordon (2010)). Moreover, even if there are multiple equilibria, we may still compare the performance of communication arising from symmetric equilibria. We state our next result in this more conservative interpretation.

Proposition 8 *Consider two symmetric distributions F, G with support on $\mathbb{R}$ and with the same finite variance σ^2. Suppose that the density f_+ is logconcave and the density g_+ is logconvex. Then, $\frac{f_+}{g_+}$ satisfies UCV+ and F_+ and G_+ satisfy CTO+. Moreover, in any informative symmetric equilibrium, there exists $a' \in (0,1)$, such that for $a \leq a'$, the distribution of μ_f^n is a strict mean preserving spread of the distribution of μ_g^n, implying that*

$$var_f(\mu_f^n) > var_g(\mu_g^n).$$

The payoffs of the communication games are thus higher for distributions with log-concave relative to logconvex densities. Note that there always exists an informative symmetric equilibrium, since binary communication is always feasible.

To prove the proposition, we only need to apply Proposition 4. Hence, we aim at showing that the conditions stated in the proposition imply that the distributions are ranked according to CTO+ and UCV+. We first consider the UCV+ order and relate it to the well-known concept of *relative logconcavity.*

Definition 4 *(Whitt (1985)) If $\frac{f_+}{g_+}$ is logconcave then f_+ is said to be logconcave relative to g_+.*

Lemma 5 *Consider two symmetric distributions with the same variance and with densities f, g on $\mathbb{R}$ such that $\frac{f_+}{g_+}$ is logconcave. Then $\frac{f_+}{g_+}$ satisfies UCV+, i.e., $g_+(\theta)$ is uniformly more variable than $f_+(\theta)$.*[22]

For example, note that $\frac{f_+}{g_+}$ is logconcave if f_+ is logconcave and g_+ is logconvex. As a consequence, two distributions satisfying the assumptions in Proposition 8 are ranked according to UCV+.

[22]From Shaked and Shanthikumar (2007) Theorem 3.A.54, it is known that relative logconcavity plus the densities crossing twice implies the uniform variability order. In contrast, we show that the uniform variability order arises from relative logconcavity plus the distributions having the same variance.

We next, consider CTO+.

Lemma 6 *Consider two symmetric densities f, g on $\mathbb{R}$. If f_+ is logconcave and g_+ is logconvex, then F_+ and G_+ satisfy CTO+, i.e., $G_+^{-1}F_+(\theta)$ is convex.*

Thus the conditions of Proposition 4 are satisfied and we obtain a mean-variance-preserving spread in terms of the underlying state distributions. By the now familiar arguments, this induces a mean-preserving spread in the distributions of receiver actions.

In closing, we note that we have picked the most focal point of comparison: the loglinear (Laplace) distribution which separates logconcave from logconvex distributions. We can pick any other distribution as a point of reference, for example the Gauss distribution: distributions that are logconcave relative to the Gauss distribution are called *strongly logconcave* (Wellner (2013)). In a similar vein, we can consider the distributions that precede or are higher than the Gauss distribution in the convex transform order. Our insights carry over to these comparisons.

All of our results indicate that communication tends to perform poorly in environments with heavier tails compared to environments with thinner tails.

7 Conclusions

In this paper, we study the impact of risk through comparative statics of distributions on the performance of communication. In particular, we are interested in the likelihood of extreme events which is tied to the likelihood of extreme disagreement in our model. We compare the payoffs under communication with those under simple delegation. We first look at an increase of risk through linear transformations of the state random variable. This amounts to changes in variance. We find that an increase of the variance scales the payoffs under communication as well as under delegation down. As a consequence, this can never imply a change of the optimal decision protocol.

We then consider an increase of risk through convex transformations of the state random variable. The convex transform order ranks equilibria in the quantile space.

In combination with a scaled version of the monotone likelihood ratio order, we rank the payoff gains from communication for distributions with bounded supports. In combination with the uniform conditional variability order, we rank the payoff gains from communication for distributions with any support, assuming adaptation costs of some size for the receiver. We find that increasing risk in terms of convex transformations decreases the gains under communication. These transformations do not impact the delegation payoff, hence delegation relative to communication becomes more often optimal.

We confirm our finding that an increase in risk is detrimental for communication when comparing distributions with thin tails with distributions with heavier tails. When extreme events become more likely, communication payoffs suffer.

Catastrophes in this paper correspond to extremely low payoffs arising from inadequate decisions. Such catastrophes are human made and stem from large disagreement in situations in which there is a good chance of an extreme state to realize. There are other situations. Suppose a catastrophe is commonly anticipated, such as a hurricane is about to hit. Arguably, such a situation can perfectly align the interests. While this is a relevant scenario, we focus on human made catastrophes in this paper leaving other cases to future work.

A Appendix

Definition A.1 *The forward equation is recursively defined as solutions* $t_{i+1}(t_{i-1}, t_i)$ *to the indifference conditions of types* t_i. *We denote an arbitrary initial value of* t_1 *by* τ. *In particular, for* $i = 1$ *we have* $t_2(0, \tau)$ *as solution to*

$$2\tau - a\mathbb{E}\left[\Theta\middle|\, \Theta \in [0, \tau]\right] - a\mathbb{E}\left[\Theta\middle|\, \Theta \in [\tau, t_2(0, \tau)]\right] = 0, \tag{5}$$

for $i > 1$ *we have* $t_{i+1}(t_{i-1}, t_i)$ *as solutions to*

$$2t_i - a\mathbb{E}\left[\Theta\middle|\, \Theta \in [t_{i-1}, t_i]\right] - a\mathbb{E}\left[\Theta\middle|\, \Theta \in [t_i, t_{i+1}(t_{i-1}, t_i)]\right] = 0. \tag{6}$$

Lemma A.1 (Szalay (2012)) *(Strict) Logconcavity of the distribution implies that*

$$\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta\middle|\, \Theta \in [t_{i-1}, t_i]\right] + \frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta\middle|\, \Theta \in [t_{i-1}, t_i]\right] \leq (<) 1.$$

Lemma A.2 *Consider the forward equation. Logconcavity of the distribution and* $a < 1$ *implies that for all* $i = 1, \ldots, n-1$

$$\frac{dt_{i+1}}{dt_i} = \frac{\left(2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta\middle|\, \Theta \in [t_{i-1}, t_i]\right] - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta\middle|\, \Theta \in [t_i, t_{i+1}]\right]\right)}{a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta\middle|\, \Theta \in [t_i, t_{i+1}]\right]} > 1.$$

Proof of Lemma A.2. Consider the forward equation for t_2. The value $t_2(0, \tau)$ is the unique solution to (5). Totally differentiating (5) we find

$$\frac{dt_2}{d\tau} = \frac{\left(2 - a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta\middle|\, \Theta \in [0, \tau]\right] - a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta\middle|\, \Theta \in [\tau, t_2]\right]\right)}{a\frac{\partial}{\partial t_2}\mathbb{E}\left[\Theta\middle|\, \Theta \in [\tau, t_2]\right]} > 1,$$

where the inequality follows from Lemma A.1:

$$2 - a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta\middle|\, \Theta \in [0, \tau]\right] > 1 > a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta\middle|\, \Theta \in [\tau, t_2]\right] + a\frac{\partial}{\partial t_2}\mathbb{E}\left[\Theta\middle|\, \Theta \in [\tau, t_2]\right].$$

Next, consider arbitrary $i = 1, \ldots, n-1$. The sender's solution to the forward equation for t_i is given by (6). Totally differentiating (6) yields

$$\left(2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right] - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta|\,\Theta \in [t_i, t_{i+1}]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right]\frac{dt_{i-1}}{dt_i}\right) dt_i$$
$$= a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta|\,\Theta \in [t_i, t_{i+1}]\right] dt_{i+1}.$$

Suppose as an inductive hypothesis that $\frac{dt_i}{dt_{i-1}} > 1$, so $\frac{dt_{i-1}}{dt_i} < 1$. Rearranging, we get

$$\begin{aligned}\frac{dt_{i+1}}{dt_i} &= \frac{\left(2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right] - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta|\,\Theta \in [t_i, t_{i+1}]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right]\frac{dt_{i-1}}{dt_i}\right)}{a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta|\,\Theta \in [t_i, t_{i+1}]\right]} \\ &> 1,\end{aligned}$$

which obtains by the inductive hypothesis and Lemma A.1:

$$\begin{aligned} & 2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right]\frac{dt_{i-1}}{dt_i} \\ > \;& 2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta|\,\Theta \in [t_{i-1}, t_i]\right] \\ > \;& 1 > a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta|\,\Theta \in [t_i, t_{i+1}]\right] + a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta|\,\Theta \in [t_i, t_{i+1}]\right].\end{aligned}$$

□

Lemma A.3 *The last equilibrium threshold t_n^n is bounded from above for all n and $\lim_{n\to\infty} t_n^n < \infty$.*

Proof of Lemma A.3. The statement is trivial for $\overline{\mathcal{S}} < \infty$.

Consider the closure condition and define

$$\Delta_n(\tau) \equiv 2t_n(\tau) - a\mathbb{E}\left[\theta|\,\theta \in [t_{n-1}(\tau), t_n(\tau)]\right] - a\mathbb{E}\left[\theta|\,\theta \geq t_n(\tau)\right].$$

Now, $\Delta_n(\tau) = 0$, for $\tau = t_1^n$. We have

$$\Delta_n(t_1^n) = 2t_n^n - a\mathbb{E}\left[\theta|\,\theta \in \left[t_{n-1}^n, t_n^n\right]\right] - a\mathbb{E}\left[\theta|\,\theta \geq t_n^n\right] \geq 2\left(t_n^n - a\mathbb{E}\left[\theta|\,\theta \geq t_n^n\right]\right),$$

which follows from $-a\mathbb{E}\left[\theta|\,\theta \in \left[t_{n-1}^n, t_n^n\right]\right] \geq -a\mathbb{E}\left[\theta|\,\theta \geq t_n^n\right]$. For a logconcave distribution, $t - a\mathbb{E}[\theta|\,\theta \geq t]$ is negative for $t = 0$, increasing in t, and goes to ∞ for $t \to \infty$. Therefore, $\lim_{n\to\infty} t_n^n < \infty$ and the sequence t_n^n is bounded above. □

Proof of Proposition 1. The proof is analogous to the proof of Proposition 1 in Deimen and Szalay (2019) which only considers the Laplace distribution, and therefore omitted. Instead of using the functional form of the Laplace distribution one can apply properties of logconcave densities to show the statements. These properties are summarized in Lemma A.1 and Lemma A.2. Moreover, Lemma A.3 shows the existence of a bound. For a detailed version of the proof, we refer the interested reader to the working paper Deimen and Szalay (2023). □

Proof of Lemma 1. Since $\mathbb{E}[\mu^n] = \mathbb{E}[\theta] = 0$ and $\mathbb{E}[\mu^n\Theta] = \mathbb{E}\mathbb{E}[\mu^n\Theta|\,\Theta \in [\theta_i, \theta_{i+1}]] = \mathbb{E}[(\mu^n)^2] = var(\mu^n)$, we have

$$\begin{aligned}\mathbb{E}u_R^{com}(y_R, \Theta, a) &= -\mathbb{E}\left[(a\mu^n - a\Theta)^2\right] = -a^2\mathbb{E}\left[(\mu^n)^2 - 2\mu^n\Theta + \Theta^2\right] \\ &= a^2\left(var(\mu^n) - \sigma^2\right).\end{aligned}$$

We now show that $var(\mu^n) = \ell(a,n)\sigma^2$, for some function $\ell(a,n)$ that is independent of σ^2. Consider a typical equilibrium indifference condition

$$t_i - a\mathbb{E}[\Theta|\,\Theta \in [t_{i-1}, t_i]] = a\mathbb{E}[\Theta|\,\Theta \in [t_i, t_{i+1}]] - t_i.$$

A change of variables to $z = \frac{\theta}{\sigma}$, and thus $dz = \frac{1}{\sigma}d\theta$, results in

$$\mathbb{E}[\Theta|\,\Theta \in [t_{i-1}, t_i]] = \frac{\int_{t_{i-1}}^{t_i} \theta\kappa\frac{1}{\sigma}\psi\left(\frac{\theta^2}{\sigma^2}\right)d\theta}{\Pr(\Theta \in [t_{i-1}, t_i])} = \frac{\sigma\int_{z_{i-1}}^{z_i} z\kappa\psi(z^2)\,dz}{\Pr(Z \in [z_{i-1}, z_i])} = \sigma\mathbb{E}[Z|\,\Theta \in [z_{i-1}, z_i]],$$

with $z_i = \frac{t_i}{\sigma}$. Hence, the indifference condition can be written as

$$z_i - a\mathbb{E}[Z|\,Z \in [z_{i-1}, z_i]] = a\mathbb{E}[Z|\,Z \in [z_i, z_{i+1}]] - z_i,$$

which is independent of the variance. As a consequence, the standardized equilibrium thresholds z_i are independent of the variance.

It follows that $var(\mu^n)$ is linear in σ^2, $var(\mu^n) = \ell(n,a)\sigma^2$, where $\ell(n,a)$ is independent of σ^2. $\square$

Proof of Lemma 2. $\mathbb{E}u_R^{del}(y_S, \Theta, a) = \mathbb{E}\left[-(\Theta - a\Theta)^2\right] = -(1-a)^2\sigma^2.$ $\square$

Proof of Proposition 2. The proof proceeds in three steps. In Step a), we compare a partition in the quantile space under distribution f to the same partition in the quantile space under distribution g. We start with $a = 1$ and then extend the comparison to $0 < a < 1$. In Step b), we consider a (partial) quantile partition which features a combination of f and g. In Step c), we combine Steps a) and b) and use an iterative procedure to derive an equilibrium partition out of the (partial) partition. This allows us to rank the equilibrium quantiles under f and g.

Step a) Let $h = f_+, g_+$ and $H = F_+, G_+$. As in Jewitt (1989) by a change of variables, the conditional expectation can be rewritten as

$$\mu_{i+1} = \mathbb{E}[\Theta|\Theta \in (t_i, t_{i+1})] = \int_{t_i}^{t_{i+1}} \theta \frac{h(\theta)}{H(t_{i+1}) - H(t_i)} d\theta = \int_{u_i}^{u_{i+1}} \frac{H^{-1}(z)}{u_{i+1} - u_i} dz$$

with $u_{i+1} = H(t_{i+1})$ and $u_i = H(t_i)$.

Define $Q_H(u_i, u_{i+1}) := H(\mu_{i+1}) = H\left(\mathbb{E}\left[\Theta|H^{-1}(u_i) \leq \Theta \leq H^{-1}(u_{i+1})\right]\right)$.

Claim. *The convex transform order implies an order of the quantiles of the conditional expectations: If $G_+^{-1}F_+(\theta)$ is convex, then $Q_{F_+}(u_i, u_{i+1}) \leq Q_{G_+}(u_i, u_{i+1})$ for all u_i, u_{i+1}, $u_i \leq u_{i+1}$, $i = 1, \ldots, n-1$.*

Proof. Assume $G_+^{-1}F_+(\theta)$ is convex. Jensen's inequality implies

$$G_+^{-1}F_+\left(\int_{u_i}^{u_{i+1}} F_+^{-1}(z) \frac{1}{F_+(F_+^{-1}(u_{i+1})) - F_+(F_+^{-1}(u_i))} dz\right)$$

$$\leq \int_{u_i}^{u_{i+1}} G_+^{-1}F_+F_+^{-1}(z) \frac{1}{u_{i+1} - u_i} dz = \int_{u_i}^{u_{i+1}} G_+^{-1}(z) \frac{1}{u_{i+1} - u_i} dz.$$

Monotonicity of G_+ implies that

$$F_+\left(\int_{u_i}^{u_{i+1}} F_+^{-1}(z) \frac{1}{u_{i+1} - u_i} dz\right) \leq G_+\left(\int_{u_i}^{u_{i+1}} G_+^{-1}(z) \frac{1}{u_{i+1} - u_i} dz\right).$$

This is equivalent to

$$F_{+}\left(\mathbb{E}\left[\Theta|F_{+}^{-1}(u_{i})\leq\Theta\leq F_{+}^{-1}(u_{i+1})\right]\right)\leq G_{+}\left(\mathbb{E}\left[\Theta|G_{+}^{-1}(u_{i})\leq\Theta\leq G_{+}^{-1}(u_{i+1})\right]\right).$$

Thus, $Q_{F_{+}}(u_{i},u_{i+1})\leq Q_{G_{+}}(u_{i},u_{i+1})$. □

Recall that the equilibrium thresholds satisfy $t_{i}^{n}-a\cdot\mu_{i}^{n}=a\cdot\mu_{i+1}^{n}-t_{i}^{n}$, for $i=1,\ldots,n$. This can be written as $t_{i}^{n}=\frac{a}{2}\cdot(\mu_{i}^{n}+\mu_{i+1}^{n})$. For now, take $a=1$.

Applying Jensen's inequality twice, we obtain

$$\begin{aligned}
&G_{+}^{-1}F_{+}\left(\frac{1}{2}\int_{u_{i-1}}^{u_{i}}\frac{F_{+}^{-1}(z)}{F_{+}(F_{+}^{-1}(u_{i}))-F_{+}(F_{+}^{-1}(u_{i-1}))}dz+\frac{1}{2}\int_{u_{i}}^{u_{i+1}}\frac{F_{+}^{-1}(z)}{F_{+}(F_{+}^{-1}(u_{i+1})-F_{+}(F_{+}^{-1}(u_{i})}dz\right)\\
&\leq\frac{1}{2}G_{+}^{-1}F_{+}\left(\int_{u_{i-1}}^{u_{i}}F_{+}^{-1}(z)\frac{1}{u_{i}-u_{i-1}}dz\right)+\frac{1}{2}G_{+}^{-1}F_{+}\left(\int_{u_{i}}^{u_{i+1}}F_{+}^{-1}(z)\frac{1}{u_{i+1}-u_{i}}dz\right)\\
&\leq\frac{1}{2}\int_{u_{i-1}}^{u_{i}}G_{+}^{-1}F_{+}F_{+}^{-1}(z)\frac{1}{u_{i}-u_{i-1}}dz+\frac{1}{2}\int_{u_{i}}^{u_{i+1}}G_{+}^{-1}F_{+}F_{+}^{-1}(z)\frac{1}{u_{i+1}-u_{i}}dz\\
&=\frac{1}{2}\int_{u_{i-1}}^{u_{i}}G_{+}^{-1}(z)\frac{1}{u_{i}-u_{i-1}}dz+\frac{1}{2}\int_{u_{i}}^{u_{i+1}}G_{+}^{-1}(z)\frac{1}{u_{i+1}-u_{i}}dz.
\end{aligned}$$

Hence

$$\begin{aligned}
&F_{+}\left(\frac{1}{2}\int_{u_{i-1}}^{u_{i}}F_{+}^{-1}(z)\frac{1}{u_{i}-u_{i-1}}dz+\frac{1}{2}\int_{u_{i}}^{u_{i+1}}F_{+}^{-1}(z)\frac{1}{u_{i+1}-u_{i}}dz\right)\\
\leq\ &G_{+}\left(\frac{1}{2}\int_{u_{i-1}}^{u_{i}}G_{+}^{-1}(z)\frac{1}{u_{i}-u_{i-1}}dz+\frac{1}{2}\int_{u_{i}}^{u_{i+1}}G_{+}^{-1}(z)\frac{1}{u_{i+1}-u_{i}}dz\right).
\end{aligned}$$

Define the functions $v\left(u_{i}\right):=\frac{1}{2}\left(\frac{1}{u_{i}-u_{i-1}}\int\limits_{u_{i-1}}^{u_{i}}F_{+}^{-1}\left(z\right)dz+\frac{1}{u_{i+1}-u_{i}}\int\limits_{u_{i}}^{u_{i+1}}F_{+}^{-1}\left(z\right)dz\right)$ and $z\left(u_{i}\right):=\frac{1}{2}\left(\frac{1}{u_{i}-u_{i-1}}\int\limits_{u_{i-1}}^{u_{i}}G_{+}^{-1}\left(z\right)dz+\frac{1}{u_{i+1}-u_{i}}\int\limits_{u_{i}}^{u_{i+1}}G_{+}^{-1}\left(z\right)dz\right)$.

Then the inequality can be written as

$$G_{+}\left(\frac{z\left(u_{i}\right)}{v\left(u_{i}\right)}v\left(u_{i}\right)\right)\geq F_{+}\left(v\left(u_{i}\right)\right)\text{ for all }u_{i}\in\left[u_{i-1},u_{i+1}\right].$$

Applying the inverse of G^{-1} and dividing by $v(u_i)$, this is equivalent to

$$\frac{z(u_i)}{v(u_i)} \geq \frac{G_+^{-1}F_+(v(u_i))}{v(u_i)} \text{ for all } u_i \in [u_{i-1}, u_{i+1}]. \tag{7}$$

We next want to introduce $a \in (0,1)$. We aim at showing that

$$G_+\left(a\frac{z(u_i)}{v(u_i)}v(u_i)\right) \geq F_+(av(u_i)) \text{ for all } a \in (0,1) \text{ and all } u_i \in [u_{i-1}, u_{i+1}].$$

Applying the inverse of G^{-1} and dividing by $av(u_i)$, this is equivalent to

$$\frac{z(u_i)}{v(u_i)} \geq \frac{G_+^{-1}F_+(av(u_i))}{av(u_i)}.$$

This is equivalent to (7) for $a = 1$. Moreover, note that the convex transform order implies the star order (see Shaked and Shanthikumar (2007), p. 214): $G_+^{-1}F_+(\theta)$ convex implies that $\frac{G_+^{-1}F_+(\theta)}{\theta}$ increases in θ.

To apply this order to our condition, note that $av(u)$ increases in a and ranges from 0 to $v(u_i)$ for $a \in [0,1]$. Hence, setting $a < 1$ reduces the value of the right side of the inequality, and since the inequality holds for $a = 1$, it continues to hold for $a < 1$.

Step b) Recall that in the quantile space, the equilibrium condition $t_i^n = \frac{a}{2} \cdot (\mu_i^n + \mu_{i+1}^n)$ can be written as

$$u_{i,h} = H_+\left(\frac{a}{2}\left(\frac{1}{u_{i,h} - u_{i-1,h}}\int\limits_{u_{i-1,h}}^{u_{i,h}} H_+^{-1}(z)\,dz + \frac{1}{u_{i+1,h} - u_{i,h}}\int\limits_{u_{i,h}}^{u_{i+1,h}} H_+^{-1}(z)\,dz\right)\right),$$

for $h = f, g$ and $H = F, G$.

Fix the equilibrium thresholds $u_{i-1,f}$ and $u_{i+1,f}$, and consider $u_i = u_{i,gf}$ as the following function that combines F and G

$$G_+\left(\frac{a}{2}\left(\frac{1}{u_i - u_{i-1,f}}\int\limits_{u_{i-1,f}}^{u_i} G_+^{-1}(z)\,dz + \frac{1}{u_{i+1,f} - u_i}\int\limits_{u_i}^{u_{i+1,f}} G_+^{-1}(z)\,dz\right)\right).$$

By Jensen's inequality, we have

$$
\begin{aligned}
& G_{+}\left(\frac{1}{2}\left(\frac{1}{u_i-u_{i-1,f}}\int\limits_{u_{i-1,f}}^{u_i} G_{+}^{-1}(z)\,dz+\frac{1}{u_{i+1,f}-u_i}\int\limits_{u_i}^{u_{i+1,f}} G_{+}^{-1}(z)\,dz\right)\right) \\
\geq \; & F_{+}\left(\frac{1}{2}\left(\frac{1}{u_i-u_{i-1,f}}\int\limits_{u_{i-1,f}}^{u_i} F_{+}^{-1}(z)\,dz+\frac{1}{u_{i+1,f}-u_i}\int\limits_{u_i}^{u_{i+1,f}} F_{+}^{-1}(z)\,dz\right)\right),
\end{aligned}
$$

for all $u_i \in [u_{i-1,f}, u_{i+1,f}]$. Thus the same inequality holds in particular at $u_{i,f}$. The fact that $\frac{G_{+}^{-1}F(\theta)}{\theta}$ is increasing in θ implies that

$$
\begin{aligned}
& G_{+}\left(\frac{a}{2}\left(\frac{1}{u_i-u_{i-1,f}}\int\limits_{u_{i-1,f}}^{u_i} G_{+}^{-1}(z)\,dz+\frac{1}{u_{i+1,f}-u_i}\int\limits_{u_i}^{u_{i+1,f}} G_{+}^{-1}(z)\,dz\right)\right) \\
\geq \; & F_{+}\left(\frac{a}{2}\left(\frac{1}{u_i-u_{i-1,f}}\int\limits_{u_{i-1,f}}^{u_i} F_{+}^{-1}(z)\,dz+\frac{1}{u_{i+1,f}-u_i}\int\limits_{u_i}^{u_{i+1,f}} F_{+}^{-1}(z)\,dz\right)\right), \qquad (8)
\end{aligned}
$$

for all $u_i \in [u_{i-1,f}, u_{i+1,f}]$.

Since condition (8) holds for any arbitrary (quantile) threshold u_i, it holds for all $i = 1, \ldots, n$.

Step c) Denote the equilibrium quantile partition under f_{+}, $u_{i,f}^{n}$, $i = 1, \ldots, n$, as

$$
u_{i,f}^{n} = F_{+}\left(\frac{a}{2}\left(\frac{1}{u_{i,f}^{n}-u_{i-1,f}^{n}}\int\limits_{u_{i-1,f}^{n}}^{u_{i,f}^{n}} F_{+}^{-1}(z)\,dz+\frac{1}{u_{i+1,f}^{n}-u_{i,f}^{n}}\int\limits_{u_{i,f}^{n}}^{u_{i+1,f}^{n}} F_{+}^{-1}(z)\,dz\right)\right)
$$

for all $i = 1, \ldots, n$. By convention, $u_{0,f}^{n} = 0$ and $u_{n+1,f}^{n} = 1$.

By Steps a) and b), we therefore have

$$
u_{i,f}^{n} \leq G_{+}\left(\frac{a}{2}\left(\frac{1}{u_{i,f}^{n}-u_{i-1,f}^{n}}\int\limits_{u_{i-1,f}^{n}}^{u_{i,f}^{n}} G_{+}^{-1}(z)\,dz+\frac{1}{u_{i+1,f}^{n}-u_{i,f}^{n}}\int\limits_{u_{i,f}^{n}}^{u_{i+1,f}^{n}} G_{+}^{-1}(z)\,dz\right)\right).
$$

Let $t_{i,gf} := G_{+}^{-1}\left(u_{i,f}^{n}\right)$. Then

$$t_{i,gf} \leq \frac{a}{2}\left(\mu_{i-1,g}\left(t_{i-1,gf}, t_{i,gf}\right) + \mu_{i,g}\left(t_{i,gf}, t_{i+1,gf}\right)\right). \tag{9}$$

It follows from this inequality, that for any fixed $t_{i-1,gf}$ and $t_{i+1,gf}$, the value of $t_i = t_{i,gf}$ is too low to be part of an equilibrium.

Given this observation, we consider the following iterative procedure: For any fixed $t_{i-1,gf}$, we denote the "partial equilibrium thresholds" under g by $t_{j,g}^{(*)}$ for all $j \geq i$, where the distribution is adjusted from f to g on the entire support, the equilibrium thresholds above $t_{i-1,gf}$ are adjusted to g, $t_j = t_{j,g}^{(*)}$ for $j \geq i$, but the equilibrium thresholds below $t_{i-1,gf}$ and not adjusted, $t_j = t_{j,gf}$ for $j < i$.

At iteration step one, keep all thresholds $t_i = t_{i,gf}$ for $i = 1, \ldots, n-1$ fixed and let t_n adjust to $t_{n,g}^{(*)} = t_{n,g}^{(*)}\left(t_{n-1,gf}\right)$. At $t_{n,g}^{(*)}$, the sender is indifferent under g between pooling upwards or downwards given that the receiver best replies with respect to g.

At iteration step l, keep all thresholds $t_i = t_{i,gf}$ for $i = 1, \ldots, n-l$ fixed, adjust threshold t_{n-l+1} to make the sender indifferent at $t_{n-l+1,g}^{(*)} = t_{n-l+1,g}^{(*)}\left(t_{n-l,gf}\right)$, and keep the sender indifferent at all thresholds $t_{j,g}^{(*)}$ for $j \geq n-l+2$. Note that all $t_{j,g}^{(*)}$ depend recursively on the initial value $t_{n-l,gf}$ and on their respective predecessors $t_{n-l+1,g}^{(*)}, \ldots, t_{j-1,g}^{(*)}$.

At iteration step one, we observe that by (9), for $t_n = t_{n,gf}$,

$$t_n - a\mu_{n,g}\left(t_{n-1,gf}, t_n\right) \leq a\mu_{n+1,g}\left(t_n, \overline{\mathcal{S}}_g\right) - t_n.$$

By logconcavity of the density, $\mu_{n+1,g}\left(t_n, \overline{\mathcal{S}}_g\right)$ and $\mu_{n,g}\left(t_{n-1,gf}, t_n\right)$ each increase in t_n less than one for one. Hence, there exists a unique $t_{n,g}^{(*)} \geq t_{n,gf}$ such that

$$t_{n,g}^{(*)} - a\mu_{n,g}(t_{n-1,gf}, t_{n,g}^{(*)}) = a\mu_{n+1,g}(t_{n,g}^{(*)}, \overline{\mathcal{S}}_g) - t_{n,g}^{(*)}. \tag{10}$$

Consider an arbitrary iteration step $l < n$. Suppose that all thresholds $t_{j,g}^{(*)}$ for $j = l+1, \ldots, n$ have been adjusted 'weakly upwards.'

Since increasing thresholds increases the right side of (9), the inequality continues to hold. It remains to be shown that there is a unique $t_l = t_{l,g}^{(*)}$ such that

$$\left(a\mu_{l+1,g}(t_l, t_{l+1,g}^{(*)}) - t_l\right) - \left(t_l - a\mu_{l,g}\left(t_{l-1,gf}, t_l\right)\right) = 0. \tag{11}$$

Differentiating the left side of (11) with respect to t_l, we get

$$-2 + a\frac{\partial}{\partial t_l}\mu_{l,g}\left(t_{l-1,gf}, t_l\right) + a\frac{\partial}{\partial t_l}\mu_{l+1,g}(t_l, t_{l+1,g}^{(*)}) + a\frac{\partial}{\partial t_{l+1}}\mu_{l+1,g}(t_l, t_{l+1,g}^{(*)})\frac{dt_{l+1,g}^{(*)}}{dt_l}.$$

By logconcavity, $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$ implies that this expression is negative. We show that $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$ holds by induction: Totally differentiating (10) with respect to $t_{n,g}^{(*)}$ and $t_{n-1,gf}$, we find that

$$\frac{dt_{n,g}^{(*)}}{dt_{n-1,gf}} = \frac{a\frac{\partial}{\partial t_{n-1,gf}}\mu_{n,g}\left(t_{n-1,gf}, t_{n,g}^{(*)}\right)}{2 - a\frac{\partial}{\partial t_{n,g}^{(*)}}\mu_{n,g}\left(t_{n-1,gf}, t_{n,g}^{(*)}\right) - a\frac{\partial}{\partial t_{n,g}^{(*)}}\mu_{n+1,g}\left(t_{n,g}^{(*)}, \overline{\mathcal{S}}_g\right)} \leq 1,$$

where the inequality is due to logconcavity of the density.

Next, suppose that $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$. Totally differentiating (11) we get

$$\begin{aligned}\frac{dt_{l,g}^{(*)}}{dt_{l-1,gf}} &= \frac{a\frac{\partial}{\partial t_{l-1,gf}}\mu_{l,g}(t_{l-1,gf}, t_{l,g}^{(*)})}{2 - a\frac{\partial}{\partial t_{l,g}^{(*)}}\mu_{l,g}(t_{l-1,gf}, t_{l,g}^{(*)}) - a\frac{\partial}{\partial t_{l,g}^{(*)}}\mu_{l+1,g}(t_{l,g}^{(*)}, t_{l+1,g}^{(*)}) - a\frac{\partial}{\partial t_{l+1,g}^{(*)}}\mu_{l+1,g}(t_{l,g}^{(*)}, t_{l+1,g}^{(*)})\frac{dt_{l+1,g}^{(*)}}{dt_{l,g}^{(*)}}} \\ &\leq 1,\end{aligned}$$

by logconcavity of the density and the assumption that $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$. This concludes the argument.

Switching back to quantiles, we have demonstrated $t_{i,g}^n \geq t_{i,gf} = G_+^{-1}\left(u_{i,f}^n\right)$, and hence

$$G_+\left(t_{i,g}^n\right) \geq u_{i,f}^n = F_+\left(t_{i,f}^n\right) \text{ for all } i.$$

$\square$

Proof of Proposition 3. Assume that the equilibrium partition under distribution

g, $t^n_{i,g}$, satisfies the following condition,

$$\begin{aligned}
&\mathbb{E}_{\hat{f}}\left[\Theta_{\hat{f}}\middle|\,\Theta_{\hat{f}}\in\left[t^n_{i-1,g},t^n_{i,g}\right]\right]+\mathbb{E}_{\hat{f}}\left[\Theta_{\hat{f}}\middle|\,\Theta_{\hat{f}}\in\left[t^n_{i,g},t^n_{i+1,g}\right]\right]\\
>&\mathbb{E}_{g}\left[\Theta_{g}\middle|\,\Theta_{g}\in\left[t^n_{i-1,g},t^n_{i,g}\right]\right]+\mathbb{E}_{g}\left[\Theta_{g}\middle|\,\Theta_{g}\in\left[t^n_{i,g},t^n_{i+1,g}\right]\right],
\end{aligned}\tag{12}$$

where $t^n_{n+1,g}=\overline{\mathcal{S}}_g$ and $t^n_{0,g}=0$.

A sufficient (but not necessary) assumption for condition (12) to hold is that MLRP$\hat{+}$ is satisfied. For future reference, Proposition 4 provides different sufficient conditions for (12) to hold.

Note that the monotone likelihood ratio property is preserved under truncation to an arbitrary interval $[t_{i-1},t_i]$,

$$\frac{\partial}{\partial\theta}\frac{\frac{\hat{f}_+(\theta)}{\hat{F}_+(t_i)-\hat{F}_+(t_{i-1})}}{\frac{g_+(\theta)}{G_+(t_i)-G_+(t_{i-1})}}=\frac{G_+\left(t_i\right)-G_+\left(t_{i-1}\right)}{\hat{F}_+\left(t_i\right)-\hat{F}_+\left(t_{i-1}\right)}\frac{\partial}{\partial\theta}\frac{\hat{f}_+(\theta)}{g_+(\theta)}>0.$$

It follows from (12) and the equilibrium conditions for $t^n_{i,g}$, $i=1,\dots,n$ that

$$t^n_{i,g}-a\mu_{i,\hat{f}}\left(t^n_{i-1,g},t^n_{i,g}\right)<a\mu_{i+1,\hat{f}}\left(t^n_{i,g},t^n_{i+1,g}\right)-t^n_{i,g}\text{ for all }i=1,\dots,n.\tag{13}$$

We now show that for all i, condition (12) implies that under distribution $\hat{f}$ the equilibrium critical types under distribution $\hat{f}$ are strictly higher and strictly better for the sender than the equilibrium critical types under distribution g.

Take the iterative procedure from the proof of Proposition 2 step c) with slightly adjusted notation: for any fixed $t^n_{i-1,g}$, denote the 'partial equilibrium thresholds' under $\hat{f}$ by $t^{(*)}_{j,\hat{f}}$ for all $j\geq i$, where the distribution is adjusted from g to $\hat{f}$ on the entire support, the equilibrium thresholds above $t^n_{i-1,g}$ are adjusted to $\hat{f}$, $t_j=t^{(*)}_{j,\hat{f}}$ for $j\geq i$, but the equilibrium thresholds below $t^n_{i-1,g}$ are not adjusted, $t_j=t^n_{j,g}$ for $j<i$.

At iteration step one, keep all thresholds $t_i=t^n_{i,g}$ for $i=1,\dots,n-1$ fixed at the equilibrium values under g, and adjust t_n to $t^{(*)}_{n,\hat{f}}=t^{(*)}_{n,\hat{f}}\left(t^n_{n-1,g}\right)$. At $t^{(*)}_{n,\hat{f}}$, the sender is indifferent under $\hat{f}$ between pooling upwards or downwards given that the receiver best replies with respect to $\hat{f}$ to the truncation above $t^n_{n-1,g}$. At iteration step l, keep all thresholds $t_i=t^n_{i,g}$ for $i=1,\dots,n-l$ fixed, adjust threshold t_{n-l+1} to make the sender indifferent at $t^{(*)}_{n-l+1,\hat{f}} = t^{(*)}_{n-l+1,\hat{f}}\left(t^{n}_{n-l,g}\right)$, and keep the sender indifferent at all thresholds $t^{(*)}_{j,\hat{f}}$ for $j \geq n-l+2$. Note that all $t^{(*)}_{j,\hat{f}}$ depend recursively on the initial value $t^{n}_{n-l,g}$ and on their respective predecessors $t^{(*)}_{n-l+1,\hat{f}}, \ldots, t^{(*)}_{j-1,\hat{f}}$.

By inequality (12) , and the equilibrium condition $t^{n}_{n,g} - a\mu^{n}_{n,g} = a\mu^{n}_{n+1,g} - t^{n}_{n,g}$, we know that for $t_n = t^{n}_{n,g}$

$$t_n - a\mu_{n,\hat{f}}\left(t^{n}_{n-1,g}, t_n\right) < a\mu_{n+1,\hat{f}}\left(t_n, \overline{\mathcal{S}}_{\hat{f}}\right) - t_n.$$

By logconcavity of the density, $\mu_{n+1,\hat{f}}\left(t_n, \overline{\mathcal{S}}_{\hat{f}}\right)$ and $\mu_{n,\hat{f}}\left(t^{n}_{n-1,g}, t_n\right)$ each increase in t_n less than one for one. Hence, there exists a unique $t^{(*)}_{n,\hat{f}} > t^{n}_{n,g}$ such that

$$t^{(*)}_{n,\hat{f}} - a\mu_{n,\hat{f}}(t^{n}_{n-1,g}, t^{(*)}_{n,\hat{f}}) = a\mu_{n+1,\hat{f}}(t^{(*)}_{n,\hat{f}}, \overline{\mathcal{S}}_{\hat{f}}) - t^{(*)}_{n,\hat{f}}. \tag{14}$$

Consider an arbitrary iteration step $l < n$. Suppose that all thresholds $t^{(*)}_{j,\hat{f}}$ for $j = l+1, \ldots, n$ have been adjusted 'upwards.' Again, increasing the thresholds reinforce inequality (13) . It remains to be shown that there is a unique $t_l = t^{(*)}_{l,f}$ such that

$$\left(a\mu_{l+1,\hat{f}}(t_l, t^{(*)}_{l+1,\hat{f}}) - t_l\right) - \left(t_l - a\mu_{l,\hat{f}}\left(t^{n}_{l-1,g}, t_l\right)\right) = 0. \tag{15}$$

Differentiating the left side of (15) with respect to t_l, we get

$$-2 + a\frac{\partial}{\partial t_l}\mu_{l,\hat{f}}\left(t^{n}_{l-1,g}, t_l\right) + a\frac{\partial}{\partial t_l}\mu_{l+1,\hat{f}}(t_l, t^{(*)}_{l+1,\hat{f}}) + a\frac{\partial}{\partial t^{(*)}_{l+1,\hat{f}}}\mu_{l+1,\hat{f}}(t_l, t^{(*)}_{l+1,\hat{f}})\frac{dt^{(*)}_{l+1,\hat{f}}}{dt_l}.$$

By logconcavity, $\frac{dt^{(*)}_{l+1,\hat{f}}}{dt_l} \leq 1$ implies that this expression is negative. We show that $\frac{dt^{(*)}_{l+1,\hat{f}}}{dt_l} \leq 1$ holds by induction: Totally differentiating (14) with respect to $t^{(*)}_{n,\hat{f}}$ and $t^{n}_{n-1,g}$, we find that

$$\frac{dt^{(*)}_{n,\hat{f}}}{dt^{n}_{n-1,g}} = \frac{a\frac{\partial}{\partial t^{n}_{n-1,g}}\mu_{n,\hat{f}}\left(t^{n}_{n-1,g}, t^{(*)}_{n,\hat{f}}\right)}{2 - a\frac{\partial}{\partial t^{(*)}_{n,\hat{f}}}\mu_{n,\hat{f}}\left(t^{n}_{n-1,g}, t^{(*)}_{n,\hat{f}}\right) - a\frac{\partial}{\partial t^{(*)}_{n,\hat{f}}}\mu_{n+1,\hat{f}}\left(t^{(*)}_{n,\hat{f}}, \overline{\mathcal{S}}_{\hat{f}}\right)} \leq 1,$$

where the inequality is due to logconcavity of the density.

Next, suppose that $\frac{dt^{(*)}_{l+1,\hat{f}}}{dt_l} \leq 1$. Totally differentiating (15) we get

$$\frac{dt^{(*)}_{l,\hat{f}}}{dt^n_{l-1,g}} = \frac{a\frac{\partial}{\partial t^n_{l-1,g}}\mu_{l,\hat{f}}(t^n_{l-1,g}, t^{(*)}_{l,\hat{f}})}{2 - a\frac{\partial}{\partial t^{(*)}_{l,\hat{f}}}\mu_{l,\hat{f}}(t^n_{l-1,g}, t^{(*)}_{l,\hat{f}}) - a\frac{\partial}{\partial t^{(*)}_{l,\hat{f}}}\mu_{l+1,\hat{f}}(t^{(*)}_{l,\hat{f}}, t^{(*)}_{l+1,\hat{f}}) - a\frac{\partial}{\partial t^{(*)}_{l+1,\hat{f}}}\mu_{l+1,\hat{f}}(t^{(*)}_{l,\hat{f}}, t^{(*)}_{l+1,\hat{f}})\frac{dt^{(*)}_{l+1,\hat{f}}}{dt^{(*)}_{l,\hat{f}}}}$$
$$\leq 1$$

by logconcavity of the density and the assumption that $\frac{dt^{(*)}_{l+1,\hat{f}}}{dt_l} \leq 1$. This concludes the argument.

Thus, we have $t^n_{i,\hat{f}} \geq t^n_{i,g}$ for $i = 1, \ldots, n$. By the monotone likelihood ratio condition, we have $\mu_{\hat{f}}(t^n_{i-1,\hat{f}}, t^n_{i,\hat{f}}) = \mu^n_{i,\hat{f}} \geq \mu^n_{i,g} = \mu_g\left(t^n_{i-1,g}, t^n_{i,g}\right)$ for $i = 1, \ldots, n+1$, where $t^n_{n+1,\hat{f}} = t^n_{n+1,g} = \overline{S}_g$.

By Lemma 1, thresholds and receiver actions are linear in scale, so $t^n_{i,\hat{f}} = \frac{\overline{S}_g}{\overline{S}_f}t^n_{i,f}$ for $i = 1, \ldots, n$ and $\mu^n_{i,\hat{f}} = \frac{\overline{S}_g}{\overline{S}_f}\mu^n_{i,f}$ for $i = 1, \ldots, n+1$. □

Proof of Proposition 4. To prove the proposition, we need to show that quantiles and receiver induced actions are more risky in the sense of a mean-variance-preserving spread under distribution F than under distribution G. Recall from the proof of Proposition 2 that CTO+ implies that the quantiles satisfy $F_+\left(t^n_{i,f}\right) \leq G_+\left(t^n_{i,g}\right)$ for all i. Thus, to prove the proposition, it suffices to order the receiver's induced actions as well.

As Figure 3 and the uniform conditional variability order UCV+ reveal, the local stochastic order depends on the location of the equilibrium thresholds considered. By symmetry, we focus on the positive half of the support only. For intervals below (above) the mode m, the truncated distributions under f_+ dominate (are dominated by) the truncated distributions under g_+ in the likelihood ratio order. To have some control over which order applies to which partition intervals – for example, to the first n intervals – it is helpful to establish monotonicity of equilibria in the alignment parameter a:

Lemma A.4 *For any symmetric logconcave density and for any n, the equilibrium critical types $t^n_i(a)$ and induced means $\mu^n_i(a)$ are strictly increasing in a for all i.*

Lemma A.4 is a corollary to Proposition 3.

The proof of Proposition 4, is completed through the following sequence of lemmas that show that condition (12) is satisfied.

Lemma A.5 *(Metzger and Rüschendorf (1991))*
Let $\frac{f_+(\theta)}{g_+(\theta)}$ be unimodal with interior mode m. The function $\frac{F_+(x)}{G_+(x)}$ inherits unimodality with mode $m_1 > m$, the function $\frac{(1-F_+(x))}{(1-G_+(x))}$ inherits unimodality with mode $m_2 < m$. Moreover, there exists a unique $\hat{x}$ such that $F_+(\theta) < G_+(\theta)$ for $\theta \in (0, \hat{x})$, $F_+(\hat{x}) = G_+(\hat{x})$, and $F_+(\theta) > G_+(\theta)$ for $\theta \in (\hat{x}, \infty)$.

Proof. Metzger and Rüschendorf (1991) Section 2. □

For the following lemma, since $\int\limits_x^{\overline{\mathcal{S}}_h} (1 - H_+(\theta))\, d\theta = \int\limits_x^{\infty} (1 - H_+(\theta))\, d\theta$ as $H_+(\theta) = 1$ for $\theta \geq \overline{\mathcal{S}}_h$, we unify notation and write $\int_x^\infty$ for infinite as well as for finite supports, $[0, \overline{\mathcal{S}}_h]$.

Lemma A.6 *(i) Let m denote the mode of the function $\frac{f_+(\theta)}{g_+(\theta)}$. Conditional on $\theta \in [0, m)$, the distributions f_+ and g_+ satisfy the monotone likelihood ratio property.*

(ii) The function $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ is unimodal in $x \in [0, \overline{\mathcal{S}}_f]$ with mode $m' \in (0, m_2)$; for $0 \leq x \leq (<)\, m'$, we have $\mathbb{E}\left[\Theta_f \middle| \Theta_f \geq x\right] \geq (>)\, \mathbb{E}\left[\Theta_g \middle| \Theta_g \geq x\right]$.

Proof of Lemma A.6. (i) Follows from the proof of Lemma 5.

(ii) We first show that $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ is unimodal with mode m'. We then show that the mode m' is interior.

Straightforward differentiation gives

$$\frac{\partial}{\partial x} \frac{\int\limits_x^\infty (1 - F_+(\theta))\, d\theta}{\int\limits_x^\infty (1 - G_+(\theta))\, d\theta} = \frac{-(1 - F_+(x)) \int\limits_x^\infty (1 - G_+(\theta))\, d\theta + (1 - G_+(x)) \int\limits_x^\infty (1 - F_+(\theta))\, d\theta}{\left(\int\limits_x^\infty (1 - G_+(\theta))\, d\theta \right)^2}.$$

The sign of the derivative is positive if and only if

$$(1 - F_+(x)) \int\limits_x^\infty (1 - G_+(\theta))\, d\theta < (1 - G_+(x)) \int\limits_x^\infty (1 - F_+(\theta))\, d\theta.$$

Note that by an integration by parts for any $x \in [0, \overline{\mathcal{S}}_h)$, we have that for $h_+ \in \{f_+, g_+\}$ and $H_+ \in \{F_+, G_+\}$

$$\mathbb{E}\left[\Theta \middle| \Theta \geq x\right] = \frac{\int\limits_x^\infty \theta h_+(\theta)\, d\theta}{1 - H_+(x)} = x + \frac{\int\limits_x^\infty (1 - H_+(\theta))\, d\theta}{1 - H_+(x)}.$$

Hence, $\frac{\partial}{\partial x} \frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta} \gtreqless 0$ if and only if $\mathbb{E}\left[\Theta_f \middle| \Theta_f \geq x\right] \gtreqless \mathbb{E}\left[\Theta_g \middle| \Theta_g \geq x\right]$.

Since a mode is an extremum, it is either at the boundary or satisfies the first order condition $\mathbb{E}\left[\Theta_f \middle| \Theta_f \geq x^*\right] = \mathbb{E}\left[\Theta_g \middle| \Theta_g \geq x^*\right]$. We next prove that there is at most one such value $x^* = m'$.

By Lemma A.5, the function $\frac{(1-F_+(x))}{(1-G_+(x))}$ is unimodal with mode m_2. Thus for $x \geq m_2$ the function is decreasing, equivalent to the conditional distribution of Θ_g conditional on $\Theta_g \geq x$ under distribution G_+ first order stochastically dominating the conditional distribution of Θ_f conditional on $\Theta_f \geq x$ under F_+: for $x \geq m_2$,

$$\frac{1 - F_+(x)}{1 - G_+(x)} > \frac{1 - F_+(\theta)}{1 - G_+(\theta)} \quad \Leftrightarrow \quad \frac{F_+(\theta) - F_+(x)}{1 - F_+(x)} > \frac{G_+(\theta) - G_+(x)}{1 - G_+(x)}.$$

By implication, for $x \geq m_2$ we have $\mathbb{E}\left[\Theta_f \middle| \Theta_f \geq x\right] < \mathbb{E}\left[\Theta_g \middle| \Theta_g \geq x\right]$ and $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ is strictly decreasing.

For $x^* < m_2$, recall that by the first order condition we have

$$-\left(1 - F_+(x^*)\right) \int\limits_{x^*}^\infty \left(1 - G_+(\theta)\right) d\theta + \left(1 - G_+(x^*)\right) \int\limits_{x^*}^\infty \left(1 - F_+(\theta)\right) d\theta = 0.$$

Differentiating a second time and evaluating at x^*, we get

$$\begin{aligned} & f_+(x^*) \int\limits_{x^*}^\infty \left(1 - G_+(\theta)\right) d\theta - g_+(x^*) \int\limits_{x^*}^\infty \left(1 - F_+(\theta)\right) d\theta \\ < \quad & g_+(x^*) \frac{1 - F_+(x)}{(1 - G_+(x))} \int\limits_{x^*}^\infty \left(1 - G_+(\theta)\right) d\theta - g_+(x^*) \int\limits_{x^*}^\infty \left(1 - F_+(\theta)\right) d\theta = 0, \end{aligned}$$

where the equality follows from the first order condition. For the inequality note that the function $\frac{(1-F_+(x))}{(1-G_+(x))}$ is increasing if and only if the hazard rates of the distributions satisfy

$$\frac{f_+(x)}{1-F_+(x)} < \frac{g_+(x)}{(1-G_+(x))},$$

thus for $x < m_2$. The second derivative being negative implies that any stationary point must be a maximum, hence there is at most one such point m'.

Finally, we prove that the mode m' of $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ must be interior. For contradiction suppose that m' is at the boundary. From the first part of the proof, $m' \leq m_2$, so that m' cannot be at the upper end of the support. Thus suppose that $m' = 0$, so that $\frac{\partial}{\partial x}\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta} < 0$ for all $x \in [0, \overline{\mathcal{S}}_f]$.

The variance of the distribution over the whole support (positive and negative) can by symmetry ($h_+ = 2h$) and by integrating by parts twice be written as

$$\int_{-\infty}^{\infty} \theta^2 h(\theta)\, d\theta = \int_0^\infty \theta^2 h_+(\theta)\, d\theta = 2\int_0^\infty \theta\,(1-H_+(\theta))\, d\theta = 2\int_0^\infty\int_x^\infty (1-H_+(\theta))\, d\theta dx,$$

with $h \in \{f, g\}$, $h_+ \in \{f_+, g_+\}$, and $H_+ \in \{F_+, G_+\}$.

We can further rewrite and integrate by parts to obtain

$$\begin{aligned}
2\int_0^\infty\int_x^\infty (1-F_+(\theta))\, d\theta dx &= 2\int_0^\infty \frac{\int_x^\infty (1-F_+(\theta))\, d\theta}{\int_x^\infty (1-G_+(\theta))\, d\theta}\int_x^\infty (1-G_+(\theta))\, d\theta dx \\
&= -2\frac{\int_z^\infty (1-F_+(\theta))\, d\theta}{\int_z^\infty (1-G_+(\theta))\, d\theta}\int_z^\infty\int_x^\infty (1-G_+(\theta))\, d\theta dx\Bigg|_0^\infty + 2\int_0^\infty \frac{\partial}{\partial z}\frac{\int_z^\infty (1-F_+(\theta))\, d\theta}{\int_z^\infty (1-G_+(\theta))\, d\theta}\int_z^\infty\int_x^\infty (1-G_+(\theta))\, d\theta dx dz \\
&= 2\frac{\int_0^\infty (1-F_+(\theta))\, d\theta}{\int_0^\infty (1-G_+(\theta))\, d\theta}\int_0^\infty\int_x^\infty (1-G_+(\theta))\, d\theta dx + 2\int_0^\infty \frac{\partial}{\partial z}\frac{\int_z^\infty (1-F_+(\theta))\, d\theta}{\int_z^\infty (1-G_+(\theta))\, d\theta}\int_z^\infty\int_x^\infty (1-G_+(\theta))\, d\theta dx dz
\end{aligned}$$

Substituting for $\mu_{h_+} = \int_0^\infty (1-H(\theta))\, d\theta$ and $\sigma_h^2 = 2\int_0^\infty\int_x^\infty (1-H_+(\theta))\, d\theta dx$, we have that

$$\sigma_f^2 - \frac{\mu_{f+}}{\mu_{g+}}\sigma_g^2 = 2\int_0^\infty \frac{\partial}{\partial z}\frac{\int_z^\infty (1-F_+(\theta))\,d\theta}{\int_z^\infty (1-G_+(\theta))\,d\theta}\int_z^\infty\int_x^\infty (1-G_+(\theta))\,d\theta dx dz.$$

We have that $m' = 0$ implies $\frac{\mu_{f+}}{\mu_{g+}} \leq 1$. Moreover, by assumption $\sigma_f^2 = \sigma_g^2$. Hence the left side is non-negative. However, the right side is strictly negative due to our contradictory hypothesis that $\frac{\partial}{\partial z}\frac{\int_z^\infty (1-F_+(\theta))d\theta}{\int_z^\infty (1-G_+(\theta))d\theta} < 0$ for all $z \in [0, \overline{\mathcal{S}}_f]$. □

To complete the proof, we note that Lemma A.6 implies that (12) in the proof of Proposition 3 applies for a sufficiently low. This in turn implies that for a fixed sender partition $(t_{i,g}^n)$, the receiver's induced actions are higher under f_+ than under g_+. Hence, by the proof of Proposition 3, the equilibrium under f_+ needs to feature higher receiver equilibrium induced actions:

Lemma A.7 *For any two symmetric, logconcave densities f, g with the same variance and with truncated densities f_+, g_+ that satisfy UCV+, there exists a unique a' such that*
$\mathbb{E}\left[\Theta_f | \, \Theta_f \geq t_{n,g}^n(a')\right] = \mathbb{E}\left[\Theta_g | \Theta_g \geq t_{n,g}^n(a')\right]$. *Moreover, for $a < a'$, all $n+1$ receiver equilibrium actions under distribution f_+ are strictly higher than under g_+, $a \cdot \mu_f\left(t_{i-1,f}^n, t_{i,f}^n\right) > a \cdot \mu_g^n\left(t_{i-1,g}^n, t_{i,g}^n\right)$ for all i.*

Proof of Lemma A.7. By Lemma A.6, the tail-truncated expectation functions, $\mathbb{E}\left[\Theta_f | \, \Theta_f \geq x\right]$ and $\mathbb{E}\left[\Theta_g | \, \Theta_g \geq x\right]$, cross exactly once in the interior of the positive half of the support. The intersection is at $x = m'$, the mode of the ratio $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$. Hence, $\mathbb{E}\left[\Theta_f | \, \Theta_f \geq t_{n,g}^n(a)\right] \geq \mathbb{E}\left[\Theta_g | \, \Theta_g \geq t_{n,g}^n(a)\right]$ if and only if $t_{n,g}^n(a) \leq m'$. By Lemma A.4, $t_{n,g}^n(a)$ is strictly increasing in a, so by continuity there is a unique a' such that $t_{n,g}^n(a') = m'$ and moreover, $t_{n,g}^n(a) < m'$ for $a < a'$.

By Lemma A.6, the distributions below $t_{n,g}^n(a)$ satisfy that $f_+(\theta)/g_+(\theta)$ increasing in θ for all $\theta \leq m$ if $t_{n,g}^n(a) \leq m$. By Lemma A.6, $m' < m_2$. By Lemma A.5, $m_2 < m$. Hence, $a \leq a'$ implies that $f_+(\theta)/g_+(\theta)$ is increasing for all $\theta \leq t_{n,g}^n(a)$. Since the monotone likelihood ratio property is preserved under multiplication of a constant, the truncated distribution below $t_{n,g}^{n}(a)$ satisfies the monotone likelihood ratio property, $\frac{\partial}{\partial\theta}\frac{f_{+}(\theta)}{F_{+}\left(t_{n,g}^{n}(a)\right)}/\frac{g_{+}(\theta)}{G_{+}\left(t_{n,g}^{n}(a)\right)}>0$. More generally, the conditional distributions truncated to any interval $\left[t_{i-1,g}^{n}(a),t_{i,g}^{n}(a)\right)$ satisfy $\frac{\partial}{\partial\theta}\frac{f_{+}(\theta)}{F_{+}\left(t_{i,g}^{n}(a)\right)-F_{+}\left(t_{i-1,g}^{n}(a)\right)}/\frac{g_{+}(\theta)}{G_{+}\left(t_{i,g}^{n}(a)\right)-G_{+}\left(t_{i-1,g}^{n}(a)\right)}>0$ for $i=1,\ldots,n$. As is well known, the monotone likelihood ratio property implies the standard stochastic order (FOSD), which in turn implies that inequality (12) is satisfied for all $i=1,\ldots,n$ if we keep the partition at the equilibrium partition under g_{+}, $(t_{i,g}^{n})$. Therefore, we can apply the proof of Proposition 3 to conclude that both the equilibrium critical types and the receiver's induced actions are increased so that $\mu_{f}\left(t_{i-1,f}^{n}(a),t_{i,f}^{n}(a)\right)\geq\mu_{g}^{n}\left(t_{i-1,g}^{n}(a),t_{i,g}^{n}(a)\right)$ for $i=1,\ldots,n$ for $a\leq a'$. $\square$

$\square$

Proof of Proposition 5. The proof of the second part regarding linear tail-truncated expectations is given in Deimen and Szalay (2019). The proof of the first part extends that proof to convex tail-truncated expectations. Before proving the result by induction, we make some preliminary observations on the conditional probabilities and the tail-truncated expectation function. A more detailed version of the proof can be found in the working paper version Deimen and Szalay (2023).

For $k=2,\ldots,n$, define $\hat{p}_{k-1}$ as the probability that $\theta\in[t_{k-2},t_{k-1}]$ conditional on $\theta\geq t_{k-2}$,

$$\hat{p}_{k-1}\equiv\frac{F_{+}(t_{k-1})-F_{+}(t_{k-2})}{1-F_{+}(t_{k-2})}.$$

Accordingly, $1-\hat{p}_{k-1}=\frac{1-F_{+}(t_{k-1})}{1-F_{+}(t_{k-2})}$ is the probability that $\theta\geq t_{k-1}$, conditional on $\theta\geq t_{k-2}$. Note that $\hat{p}_{k-1}\mu_{k-1}=\mathbb{E}\left[\theta|\,\theta\geq t_{k-2}\right]-(1-\hat{p}_{k-1})\,\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}\right].$ Solving for $\hat{p}_{k-1}$, we can write the probabilities as

$$\hat{p}_{k-1}=\frac{\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}\right]-\mathbb{E}\left[\theta|\,\theta\geq t_{k-2}\right]}{\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}\right]-\mu_{k-1}}\quad\text{and}\quad 1-\hat{p}_{k-1}=\frac{\mathbb{E}\left[\theta|\,\theta\geq t_{k-2}\right]-\mu_{k-1}}{\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}\right]-\mu_{k-1}}.$$

Observe that $(1-\hat{p}_{k-2})\cdot\hat{p}_{k-1}$ is the probability of the event $\theta\in[t_{k-2},t_{k-1}]$ conditional on $\theta\geq t_{k-3}$, and $(1-\hat{p}_{k-2})\cdot(1-\hat{p}_{k-1})$ is the probability of the event $\theta\geq t_{k-1}$ conditional on $\theta\geq t_{k-3}$. To see this, note that $1-\hat{p}_{k-2}=\Pr\left[\theta\geq t_{k-2}|\,\theta\geq t_{k-3}\right]=\frac{1-F_{+}(t_{k-2})}{1-F_{+}(t_{k-3})}$ and recall that $\hat{p}_{k-1}=\frac{F_{+}(t_{k-1})-F_{+}(t_{k-2})}{1-F_{+}(t_{k-2})}$.

Define, for all $t > 0$

$$\alpha\left(t\right) := \frac{\mathbb{E}\left[\theta|\,\theta \geq t\right] - \mathbb{E}\left[\theta|\,\theta \geq 0\right]}{t} = \frac{\phi(t) - \phi(0)}{t}.$$

Define $\mu_+ := \phi(0) = \mathbb{E}\left[\theta|\,\theta \geq 0\right]$. Note that $\phi(t) = \mathbb{E}\left[\theta|\,\theta \geq t\right]$ can always be written as the *pseudo linear interpolation* $\mathbb{E}\left[\theta|\,\theta \geq t\right] = \mu_+ + t \cdot \alpha\left(t\right)$. In the case of a linear tail-truncated expectation, $\alpha(t)$ is a constant. In the convex case, we show that $\alpha(t)$ is increasing in t:

For $t = 0$, we take the limit $\alpha\left(0\right) = \lim_{t\to 0} \frac{\mathbb{E}[\theta|\theta \geq t] - \mu_+}{t} = \frac{\partial}{\partial t}\,\mathbb{E}\left[\theta|\,\theta \geq t\right]\big|_{t=0}$. Likewise, by l'Hôpital's rule, $\lim_{t\to\infty} \frac{\mathbb{E}[\theta|\theta \geq t] - \mu_+}{t} = \lim_{t\to\infty} \frac{\partial}{\partial t}\mathbb{E}\left[\theta|\,\theta \geq t\right]$. Moreover, $\alpha'\left(t\right) = \frac{\frac{\partial}{\partial t}\mathbb{E}[\theta|\theta \geq t]t - \left(\mathbb{E}[\theta|\theta \geq t] - \mu_+\right)}{t^2} = \frac{1}{t}\left(\frac{\partial}{\partial t}\mathbb{E}\left[\theta|\,\theta \geq t\right] - \alpha\left(t\right)\right)$. By the fundamental theorem of calculus $\alpha\left(t\right) = \frac{\mathbb{E}[\theta|\theta \geq t] - \mu_+}{t} = \frac{\int_0^t \frac{\partial}{\partial z}\mathbb{E}[\theta|\theta \geq z]dz}{t}$. By the intermediate value theorem for integrals, there is some value $t^* \in (0, t)$ such that $\frac{\int_0^t \frac{\partial}{\partial z}\mathbb{E}[\theta|\theta \geq z]dz}{t} = \frac{\partial}{\partial z}\mathbb{E}\left[\theta|\,\theta \geq z\right]\big|_{z=t^*}$. Hence, $\alpha'\left(t\right) = \frac{1}{t}\left(\frac{\partial}{\partial z}\mathbb{E}\left[\theta|\,\theta \geq z\right]\big|_{z=t} - \frac{\partial}{\partial z}\mathbb{E}\left[\theta|\,\theta \geq z\right]\big|_{z=t^*}\right) \geq 0$, where the inequality follows from $t^* \in (0, t)$ and from convexity of $\mathbb{E}\left[\theta|\,\theta \geq t\right]$ in t. Thus, $\alpha(t)$ is increasing in t and hence minimal at $\alpha\left(0\right) =: \underline{\alpha}$.

Recall the alignment parameter $a \in (0, 1)$. Define $\hat{c} := \underline{\alpha} \cdot a$.

Assume that $\hat{c} \in (0, 2)$. Note that for all distributions with logconcave densities this is not a constraint. In this case, $\underline{\alpha} \leq \alpha(t) \leq 1$ for all t, since logconcave densities have a decreasing mean residual life (see Bagnoli and Bergstrom (2005), Theorem 3 and Lemma 2) and $\alpha(t) > 1$ for some $t > 0$ would imply that the mean residual life at t is higher than at zero, a contradiction.

Let $X_k^n\left(t_{k-1}^n\right)$ be equal to $\hat{c}^2$ times the expected squared deviation of the truncated means from μ_+, conditional on $\theta \geq t_{k-1}^n$,

$$X_k^n\left(t_{k-1}^n\right) \;:=\; \hat{p}_k^n\left(\hat{c}\mu_k^n - \hat{c}\mu_+\right)^2 + \left(1 - \hat{p}_k^n\right)X_{k+1}^n(t_k).$$

Induction hypothesis.

$$X_k^n\left(t_{k-1}^n\right) \geq \underline{X}_k^n\left(t_{k-1}^n\right) := \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) \\ + 2\left(\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right).$$

Induction base. The proof of the induction base is a simplified version of the proof of the induction step and therefore omitted.

Induction step.

By definition, $X_{k-1}^n\left(t_{k-2}^n\right) = \hat{p}_{k-1}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \left(1-\hat{p}_{k-1}\right) X_k^n\left(t_{k-1}^n\right)$. Since $X_k^n\left(t_{k-1}^n\right) \geq \underline{X}_k^n\left(t_{k-1}^n\right)$, we have

$$X_{k-1}^n\left(t_{k-2}^n\right) \geq \hat{p}_{k-1}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \left(1-\hat{p}_{k-1}\right)\underline{X}_k^n\left(t_{k-1}^n\right) =: \widehat{X}_{k-1}^n\left(t_{k-2}^n\right).$$

Substituting for the probability $\hat{p}_{k-1}$ and for $\underline{X}_k^n\left(t_{k-1}^n\right)$, we obtain

$$\widehat{X}_{k-1}^n\left(t_{k-2}^n\right) = \frac{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-2}^n\right]}{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 \\ + \frac{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-2}\right] - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\begin{array}{c} \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) \\ +2\left(\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right) \end{array}\right).$$

Expanding the numerators of the probabilities by $\pm\hat{c}\mu_+$ and reorganizing according to common factors, we can write $\widehat{X}_{k-1}^n\left(t_{k-2}^n\right) = A_{k-1}^n + B_{k-1}^n$, with

$$A_{k-1}^n \equiv \frac{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \frac{\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n} \\ \cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) + 2\left(\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right)\right)$$

and

$$B_{k-1}^n \equiv \frac{\hat{c}\mu_+ - \hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-2}^n\right]}{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \frac{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-2}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n} \\ \cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu_k^n\right)\left(\hat{c}\mu_+ - \hat{c}\mu_k^n\right) + 2\left(\hat{c}\mathbb{E}\left[\theta|\, \theta \geq t_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_k^n\right) - \hat{c}\mu_+\right)\right).$$

The indifference condition of type t_{k-1}^n, $\hat{c}\mu_k^n = 2\underline{\alpha}t_{k-1}^n - \hat{c}\mu_{k-1}^n$, allows us to substitute for $\hat{c}\mu_k^n$. Hence,

$$\begin{aligned} A_{k-1}^n =& \frac{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)^2 + \frac{\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n} \\ &\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + 2\underline{\alpha}t_{k-1}^n - \hat{c}\mu_{k-1}^n\right)\left(\hat{c}\mu_+ - \left(2\underline{\alpha}t_{k-1}^n - \hat{c}\mu_{k-1}^n\right)\right)\right. \\ &\quad \left. +2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}\right] - \hat{c}\mu_+\right)\left(\frac{1}{2-\hat{c}}\left(\hat{c}\mu + 2\underline{\alpha}t_{k-1} - \hat{c}\mu_{k-1}^n\right) - \hat{c}\mu_+\right)\right). \end{aligned}$$

Collecting terms with the common factor $\frac{\hat{c}\mathbb{E}\left[\theta|\theta\geq t_{k-1}^n\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\theta\geq t_{k-1}^n\right]-\hat{c}\mu_{k-1}^n}\left(\hat{c}\mu_{k-1}^n - \hat{c}\mu_+\right)$ and simplifying, we get

$$\begin{aligned} A_{k-1}^n &= \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n\right)\left(\hat{c}\mu_+ + \hat{c}\mu_{k-1}^n\right) \\ &\quad + \frac{\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\begin{array}{c} \frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t_{k-1}^n\right)^2 + 4\underline{\alpha}t_{k-1}^n c\mu_{k-1}\right) \\ +\left(\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{4}{2-\hat{c}}\left(\underline{\alpha}t_{k-1}^n - \hat{c}\mu_{k-1}^n\right)\right) \end{array}\right). \end{aligned}$$

Similarly, we can derive

$$\begin{aligned} B_{k-1}^n &= 2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-2}^n\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu_{k-1}^n\right) - \hat{c}\mu_+\right) \\ &\quad + \frac{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-2}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}\left(\begin{array}{c} \frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t_{k-1}^n\right)^2 + 4\underline{\alpha}t_{k-1}^n\hat{c}\mu_{k-1}^n\right) \\ +\left(\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_+\right)\left(\frac{4}{2-\hat{c}}\underline{\alpha}t_{k-1}^n - \frac{4}{2-\hat{c}}\hat{c}\mu_{k-1}^n\right) \end{array}\right). \end{aligned}$$

We aim at showing that the second lines in A_k and B_k respectively are both positive. We then obtain a lower bound on $\widehat{X}_{k-1}^n$ by discarding them.

Note that

$$\frac{\hat{c}\mu_+ - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n} + \frac{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-2}^n\right] - \hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n} = \frac{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-2}^n\right] - \hat{c}\mu_{k-1}^n}{\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_{k-1}^n}.$$

Since $\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] > \mathbb{E}\left[\theta|\,\theta \geq t_{k-2}^n\right] > \mathbb{E}\left[\theta|\,\theta \in \left[t_{k-2}^n, t_{k-1}^n\right]\right] = \mu_{k-1}^n$, both the denominator and the numerator are positive.

By the definitions of α, $\underline{\alpha}$, and by convexity, $\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t_{k-1}^n\right] - \hat{c}\mu_+ = \hat{c}\alpha\left(t_{k-1}^n\right)t_{k-1}^n \geq \hat{c}\underline{\alpha}t_{k-1}^n$. Moreover, since $a < 1$ and $t_{k-1}^n \geq \mu_{k-1}^n$, $\underline{\alpha}t_{k-1}^n - \hat{c}\mu_{k-1}^n = \underline{\alpha}\left(t_{k-1}^n - a\mu_{k-1}^n\right) \geq 0$.

Taken together, we get

$$\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}^{n}\right]-\hat{c}\mu_{+}\right)\left(\frac{4}{2-\hat{c}}\underline{\alpha}t_{k-1}^{n}-\frac{4}{2-\hat{c}}\hat{c}\mu_{k-1}^{n}\right)\geq\hat{c}\underline{\alpha}t_{k-1}^{n}\left(\frac{4}{2-\hat{c}}\underline{\alpha}t_{k-1}^{n}-\frac{4}{2-\hat{c}}\hat{c}\mu_{k-1}^{n}\right),$$

and therefore

$$\begin{aligned}
& \frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t_{k-1}^{n}\right)^{2}+4\underline{\alpha}t_{k-1}^{n}\hat{c}\mu_{k-1}^{n}\right)+\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}^{n}\right]-\hat{c}\mu_{+}\right)\left(\frac{4}{2-\hat{c}}\underline{\alpha}t_{k-1}^{n}-\frac{4}{2-\hat{c}}\hat{c}\mu_{k-1}^{n}\right)\\
\geq\ & \frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t_{k-1}^{n}\right)^{2}+4\underline{\alpha}t_{k-1}^{n}\hat{c}\mu_{k-1}^{n}\right)+\hat{c}\underline{\alpha}t_{k-1}^{n}\left(\frac{4}{2-\hat{c}}\underline{\alpha}t_{k-1}^{n}-\frac{4}{2-\hat{c}}\hat{c}\mu_{k-1}^{n}\right)\\
=\ & 0.
\end{aligned}$$

Note that all inequalities involving $\underline{\alpha}$ are strict for the case in which $\alpha\left(t_{k-1}^{n}\right)>\underline{\alpha}$. This implies that the second lines in A_{k}^{n} and B_{k}^{n} are indeed positive. Hence, we have

$$\begin{aligned}
X_{k-1}^{n}\left(t_{k-2}^{n}\right) &\geq \widehat{X}_{k-1}^{n}\left(t_{k-2}^{n}\right)\\
&\geq \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_{+}-\hat{c}\mu_{k-1}^{n}\right)\left(\hat{c}\mu_{+}+\hat{c}\mu_{k-1}^{n}\right)\\
&\quad +2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t_{k-2}^{n}\right]-\hat{c}\mu_{+}\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_{+}+\mu_{k-1}^{n}\right)-\hat{c}\mu_{+}\right).
\end{aligned}$$

This concludes the induction step.

It follows that $X_{1}^{n}\left(t_{0}^{n}\right)\geq\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_{1}^{n}+\hat{c}\mu_{+}\right)\left(\hat{c}\mu_{+}-\hat{c}\mu_{1}^{n}\right)$.

By definition, $X_{1}^{n}\left(t_{0}^{n}\right)=\mathbb{E}\left[\left(\hat{c}\mu_{i}^{n}-\hat{c}\mu_{+}\right)^{2}\right]$. Canceling $\hat{c}$, we get

$$\mathbb{E}\left[\left(\mu_{i}^{n}-\mu_{+}\right)^{2}\right]\geq\frac{\underline{\alpha}a}{2-\underline{\alpha}a}\left(\mu_{+}^{2}-\left(\mu_{1}^{n}\right)^{2}\right)$$

with strict inequality if $\mathbb{E}\left[\theta|\,\theta\geq t\right]$ is strictly convex in t. Decentering again and noting that by the law of iterated expectations $\mathbb{E}\left[\mu_{i}^{n}\mu_{+}\right]=\mathbb{E}\left[(\mu_{+})^{2}\right]$, we can write

$$\mathbb{E}\left[(\mu_{i}^{n})^{2}\right]\geq\frac{\underline{\alpha}a}{2-\underline{\alpha}a}\left(\mu_{+}^{2}-(\mu_{1}^{n})^{2}\right)+\mu_{+}^{2}.$$

Recall that $\phi(0)=\mu_{+}$ and $\underline{\alpha}=\phi'(0)$. Thus, for limit $n\to\infty$, we have $\mu_{1}^{n}\to0$ and

$$var(\mu^{\infty})=\mathbb{E}\left[(\mu_{i}^{\infty})^{2}\right]\geq\frac{\underline{\alpha}a}{2-\underline{\alpha}a}\mu_{+}^{2}+\mu_{+}^{2}=\frac{2}{2-\phi'(0)a}\phi(0)^{2}.$$

$\square$

Proof of Lemma 3. Straightforward integration gives for any $[\underline{t}, \overline{t}] \subseteq \left[0, -\frac{s}{\delta}\right]$,

$$\mathbb{E}\left[\Theta|\Theta \in [\underline{t}, \overline{t}]\right] = \frac{s+\overline{t}}{1-\delta} - \frac{1}{1-\delta} \frac{\left(\overline{t}-\underline{t}\right)}{1-\left(\frac{1+\frac{\delta}{s}\overline{t}}{1+\frac{\delta}{s}\underline{t}}\right)^{-\frac{1}{\delta}}}. \tag{16}$$

For the special case of $\overline{t} = -\frac{s}{\delta}$ and $\underline{t} \in \left[0, -\frac{s}{\delta}\right]$, we get

$$\mathbb{E}\left[\Theta|\Theta \geq \underline{t}\right] = \mathbb{E}\left[\Theta|\Theta \geq 0\right] + \frac{1}{1-\delta} \cdot \underline{t} = \frac{s+\underline{t}}{1-\delta}. \tag{17}$$

Hence, the generalized Pareto distribution features linear tail-truncated expectations. Therefore, we can apply the value characterization of Deimen and Szalay (2019), which derives the expected utility of a limit equilibrium given in (3) as an upper bound on the expected utilities of finite equilibria. The variance of μ^n in a Even equilibrium is given by

$$var(\mu^n) = \frac{2}{2-\frac{a}{1-\delta}}\mu_+^2 - \frac{\frac{a}{1-\delta}}{2-\frac{a}{1-\delta}}\left(\mu_1^n\right)^2.$$

The variance of μ^n in an Odd equilibrium is given by

$$var(\mu^n) = \left(1 - \Pr\left[\Theta \in \left[-\frac{a\mu_2^n}{2}, \frac{a\mu_2^n}{2}\right)\right]\right) \cdot \left(\frac{2}{2-\frac{a}{1-\delta}}\mu_+^2 + \frac{\frac{a}{1-\delta}}{2-\frac{a}{1-\delta}}\mu_2^n\mu_+\right).$$

Deimen and Szalay (2019) shows that a limit equilibrium exists for the special case of $\delta = 0$. Here, we extend the proof of existence of a limit equilibrium in Proposition 1 to the class of all logconcave densities, which includes the generalized Pareto distribution with $\delta \in [-1, 0]$. $\square$

Proof of Proposition 6. One can show that our limit equilibrium yields a higher payoff than any finite equilibrium in the communication game. Compare the receiver's expected utility in a limit equilibrium under communication

$$\mathbb{E}u_R\left(a\mu^\infty,\Theta,a\right)=a^2\left(var(\mu^\infty)-\sigma^2\right)=a^2\left(\frac{2-\frac{1}{1-\delta}}{2-\frac{a}{1-\delta}}\sigma^2-\sigma^2\right)=-a^2\sigma^2\frac{1-a}{2-a-2\delta}$$

to the receiver's expected utility under delegation $\mathbb{E}u_R\left(\Theta,\Theta,a\right)=-\left(1-a\right)^2\sigma^2$. The receiver prefers delegation over communication if

$$-\left(1-a\right)^2\sigma^2\geq-a^2\sigma^2\frac{1-a}{2-a-2\delta}\quad\Leftrightarrow\quad\delta\geq\frac{2-3a}{2-2a}.$$

□

Proof of Lemma 4. Since the Gauss distribution features a convex tail-truncated expectation (see Sampford (1953)), the minimal slope for the tail-truncated expectation is obtained at $\theta=0$.

$$\left.\frac{\partial}{\partial t}\mathbb{E}\left[\Theta|\,\Theta\geq t\right]\right|_{t=0}=\left(\mathbb{E}\left[\Theta|\,\Theta\geq t\right]-t\right)\left.\frac{f\left(t\right)}{1-F\left(t\right)}\right|_{t=0}=\frac{\phi(0)}{\sigma}2\frac{1}{\sqrt{2\pi}}.$$

Moreover, we have $\left.\mathbb{E}\left[\Theta|\,\Theta\geq t\right]\right|_{t=0}=\phi(0)=\sigma\left.\frac{f(t)}{1-F(t)}\right|_{t=0}=\sigma\frac{\sqrt{2}}{\sqrt{\pi}}$. Substituting in (3) for $\phi(0)$ and the minimal slope, we obtain the result. □

Lemma A.8 *Denote the Gauss distribution by F and the Laplace distribution by G,*

i) then F and G satisfy CTO+.

ii) then F and G feature a unimodal likelihood ratio $\frac{f_+}{g_+}$.

iii) then F induces a higher value of communication than G for $a<0.858$.

Proof of Lemma A.8. i) Follows from van Zwet (1964) p.59, as the Gauss distribution has an increasing hazard rate.

ii) Let g_+ be the Laplace and f_+ be the Gauss 'half'-densities. Then

$$\frac{f_+\left(\theta\right)}{g_+\left(\theta\right)}=\frac{\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\theta^2}{2\sigma^2}}}{\frac{\sqrt{2}}{\sigma}e^{-\frac{\sqrt{2}}{\sigma}\theta}}=\frac{e^{\left(\frac{\sqrt{2}}{\sigma}\theta-\frac{\theta^2}{2\sigma^2}\right)}}{2\sqrt{\pi}},$$

and we observe that $\frac{f_+(\theta)}{g_+(\theta)}$ is increasing for low levels of θ and decreasing for high levels of θ.

iii) Comparing the values of communicating under the Gauss and Laplace distributions, we find that the Gauss distribution induces a higher value of communication than the Laplace:

$$a^2\left(\frac{\frac{4}{\pi}}{2-a\frac{2}{\pi}}\sigma^2-\sigma^2\right)\geq a^2\left(\frac{1}{2-a}\sigma^2-\sigma^2\right), \tag{18}$$

which holds for $a \lesssim 0.858$. $\square$

Proof of Proposition 7. For the Laplace distribution, communication is preferred over delegation if

$$a^2\left(\frac{1}{2-a}\sigma^2-\sigma^2\right)\geq-(1-a)^2\sigma^2, \tag{19}$$

which holds if and only if $a \leq \frac{2}{3}$. Therefore, for $a \leq \frac{2}{3}$, (19) and (18) form a chain of inequalities implying that communication is also preferred over delegation for the Gauss distribution.

If the state follows a Gauss distribution, using the lower bound for communication, we obtain that communication is preferred over delegation if

$$a^2\left(\frac{\frac{4}{\pi}}{2-a\frac{2}{\pi}}\sigma^2-\sigma^2\right)\geq-(1-a)^2\sigma^2,$$

which holds for $a \lesssim 0.702$. Hence, for $a \in \left(\frac{2}{3}, 0.702\right)$ delegation is strictly optimal if the state follows a Laplace distribution while communication is strictly optimal if the state follows a Gauss distribution. $\square$

Proof of Lemma 5. Since the supports are assumed to be $\mathbb{R}$, we have $supp(f) \subseteq supp(g)$. It remains to be shown that the ratio $\frac{f_+(\theta)}{g_+(\theta)}$ is unimodal with mode m an interior maximum.

Logconcavity of the ratio $\frac{f_+(\theta)}{g_+(\theta)}$ is equivalent to $\frac{\partial}{\partial\theta}\left(\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)}-\frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}\right)\leq 0$. That the difference is falling implies that one of three cases holds: either the difference is positive for all θ, $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} > \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$, negative for all θ, $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} < \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$, or changes sign once, i.e., there is some value m such that $\frac{\frac{\partial}{\partial\theta}f_+(\theta)|_{\theta=m}}{f_+(m)} = \frac{\frac{\partial}{\partial\theta}g_+(\theta)|_{\theta=m}}{g_+(m)}$ and $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} >$

$\frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$ for $\theta \in [0, m)$ and $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} < \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$ for $\theta \in (m, \overline{\mathcal{S}}]$.

The first two cases amount to MLRP on the positive half and can be ruled out by the following argument: Monotonicity of the likelihood ratio for all $\theta > 0$ implies that $F_+(\theta)$ and $G_+(\theta)$ are ranked in the standard stochastic order (one distribution first order stochastically dominates the other one, FOSD). By symmetry, this implies that $F(\theta)$ and $G(\theta)$ are ordered in the convex order (SOSD). Finally, this implies that the distributions must have different variances, contradicting our assumption.

Hence, case three applies, implying that $\frac{f_+}{g_+}$ is unimodal with unique interior mode m. By concavity the mode is a maximum. $\square$

Proof of Lemma 6. We show that the convex transform order CTO+ is transitive. Note that

$$G_+^{-1}F_+(\theta) = G_+^{-1}H_+H_+^{-1}F_+(\theta).$$

Since $G_+^{-1}H_+(\theta)$ and $H_+^{-1}F_+(\theta)$ are increasing functions, $G_+^{-1}F_+(\theta)$ is convex if $G_+^{-1}H_+(\theta)$ and $H_+^{-1}F_+(\theta)$ are convex.

Recall that a Laplace distribution is a two-sided exponential distribution. van Zwet (1964) shows that for H_+ the exponential distribution, $H_+^{-1}F_+(\theta)$ is convex for any distribution F_+ with an increasing hazard rate. Since logconcavity of the density implies an increasing hazard rate (Bagnoli and Bergstrom (2005)), $H_+^{-1}F_+(\theta)$ is convex. Likewise, by van Zwet (1964), $H_+^{-1}G_+(\theta)$ is concave for any distribution G_+ with a decreasing hazard rate. Again, logconvexity of the density implies a decreasing hazard rate (Bagnoli and Bergstrom (2005)).

Hence, we need to show that $H_+^{-1}G_+(\theta)$ is concave if and only if $G_+^{-1}H_+(\theta)$ is convex. We note that $H_+^{-1}G_+(\theta)$ is concave if and only if $\frac{g_+\left(G_+^{-1}(u)\right)}{h_+\left(H_+^{-1}(u)\right)}$ is decreasing in $u \in [0,1]$ while $G_+^{-1}H_+(\theta)$ is convex if and only if $\frac{h_+\left(H_+^{-1}(u)\right)}{g_+\left(G_+^{-1}(u)\right)}$ is increasing in $u \in [0,1]$. Hence, $H_+^{-1}F_+(\theta)$ is convex if and only if $H_+^{-1}G_+(\theta)$ is concave. $\square$

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