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Communication in the shadow of catastrophe ☆

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ABSTRACT

We perform distributional comparative statics in a cheap talk model of adaptation. Receiver borne adaptation costs drive a wedge between the objectives of sender and receiver that is increasing in the magnitude of adaptation. We allow for infinite supports with infinite disagreement at the extremes and compare communication to unconstrained delegation. We study increases in risk that arise from transformations of the state variable. We find that linear transformations (implying increases in variance) decrease communication and delegation payoffs but do not change their ranking. By contrast, increasing, convex transformations (implying increases in *tail risk*) decrease the communication payoff relative to the delegation payoff. Our finding extends to the comparison of distributions with thin versus heavy tails.

1. Introduction

Catastrophes, in many instances, result from a combination of extreme circumstances and insufficient responses to them. Responses are often insufficient because of costs that are borne by decision makers. Are there simple ways to mitigate catastrophic outcomes in a world in which extreme circumstances, and the significant cost of responses to them, are relatively likely? If an expert realizes that they are in the shadow of a looming catastrophe, can the expert communicate successfully to the decision maker and induce an action that is sufficient to mitigate the catastrophe?

For an illustration of unusually extreme circumstances combined with insufficient responses, consider the following three examples with catastrophic outcomes. Experts suggested adjusting drilling procedures prior to the blowout on the *Deepwater Horizon* in 2010; *BP* however decided not to change its procedures against expert advice.[1] At the time of the *Challenger* space shuttle explosion in 1986, engineers warned in vain about potential problems arising from low temperatures.[2] Officials delayed the evacuation of the *Ahr* valley in 2021 despite experts' warnings of an extreme rise of the water level and subsequent flooding.[3] The examples feature conditions

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[1] For details, see the of the National Comission on the BP Deepwater Horizon oil spill and offshore drilling (2011) and our discussion in the following subsection.

[2] See, for example, https://www.nytimes.com/2016/03/26/science/robert-ebeling-challenger-engineer-who-warned-of-disaster-dies-at-89.html.

[3] See, for example, https://www.nytimes.com/2021/07/16/world/europe/germany-floods-climate-change.html.

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of extraordinarily high pressure, extreme temperature or an exceptionally high rise in the water level, combined with substantial costs of adaptation in each case. Communication prior to these impending catastrophes did not, evidently, help to avert them; expert advice was not taken into account sufficiently.

These are examples of organizational failures, for which there are many reasons (see Garicano and Rayo (2016)). We focus on a class of such organizational failures that stem from a combination of misalignment of incentives (because of adjustment costs) and *tail risk*, an increased likelihood of extreme events.[4] In particular, we investigate three questions: How well does communication work in environments where the risk of extreme circumstances, and the costs of mitigating them, are high? How do changes in the environment impact communication? Does an increase in risk impact the decision to delegate the decision rights to an expert?

Formally, this paper studies communication between a scientific expert (sender) and a decision-maker (receiver). The sender observes the state of the world and sends a cheap talk message (Crawford and Sobel (1982)) to the receiver who then takes an action. The sender as well as the receiver care about the appropriate action being taken given the state. The receiver, however, faces additional concerns such as political costs or firm-specific costs of adaptation. Formally, sender and receiver share a common loss function. In addition to this common loss, the receiver faces a cost of adaptation. The expected optimal action requires no adaptation, so there is agreement. The costs drive a wedge between the objectives of the decision-maker and the expert; this wedge is increasing in the magnitude of adaptation. As we allow for an infinite support of the state, there is infinite disagreement at extreme states.

Our main focus is on the impact of the distributional environment on communication, in particular, of the relative likelihood of extreme states and extreme disagreement. We show that equilibrium payoffs decrease as extreme states become more likely. We provide stochastic orders that allow us to rank distributions with respect to their impact on equilibria and on payoffs. We compare the communication outcomes to the outcomes under the alternative decision protocol of unconstrained delegation, where the decision-maker delegates decision-making to the expert. We find that delegation becomes relatively more attractive when extreme conflicts become relatively more likely, i.e., when the tail risk increases.

To be more precise, we establish existence and essential uniqueness of cheap talk equilibria that induce a given number of receiver actions up to a countable infinity (Proposition 1). To study the impact of increased risk on communication, we analyze two types of transformations of the state variable. We first consider *linear* transformations that scale the state proportionally to the original state and thus increase the variance, but maintain the shape of the distribution. We show that the effect is a linear spread of the equilibrium actions, resulting in a reduction of expected utilities proportional to the increase in variance (Lemma 1). Since the payoff under delegation is also proportional to the variance, such a linear transformation never implies a switch in the optimal decision-procedure (Corollary 1).

Second, we consider combinations of linear and increasing, *convex* transformations of the state variable. The linear transformation scales the distribution and hence controls for the variance.[5] The increasing, convex transformation changes the shape of the distribution such that the tails of the distribution become heavier, hence increasing the kurtosis. In this case, we say that the distribution becomes more *tail-risky*.

Formally, we consider symmetric distributions and assume that the distributions of the absolute values of the states are ordered in the *convex transform order* (van Zwet (1964)). This implies that one distribution is more skewed towards large absolute values than the other. By symmetry, skewness on each side of the prior mean translates into a higher kurtosis of the overall distribution. Building on Jensen's inequality, we show that a more convex, tail-risky, environment implies that the equilibrium critical types are higher in the quantile space (Proposition 2).

For our comparison of receiver equilibrium actions and payoffs, we complement the convex transform order with the *uniform conditional variability order* (Whitt (1985)). The order implies a unimodal likelihood ratio: the less variable distribution of absolute values is stochastically higher for small deviations from the prior mean, while the more variable distribution is stochastically higher for large deviations from the prior mean. The two orders together imply that the densities of the absolute values cross exactly twice: in the less tail-risky environment, intermediate adjustments are more often required, whereas in the more tail-risky environment, very small and very large adjustments are more often required.

For sufficiently high marginal costs of adaptation, all the sender equilibrium critical types are close to the prior mean. In combination with the ordering of the quantiles, we show that the distribution of receiver actions in the *less* tail-risky environment is a *mean-preserving spread* of the corresponding actions in the more tail-risky environment (Proposition 3). Thus, there is *more* information transmission in the *less* tail-risky environment.

Intuitively, the likelihood ratio of the distributions is hump-shaped, so that the less tail-risky distribution features stochastically higher deviations from the prior mean for small deviations, while the more tail-risky distribution features stochastically higher deviations from the prior mean for large deviations. Increases in the marginal costs of adaptation move the receiver's actions closer to the prior mean. Thus, insufficient responses to outliers are the reason why the more tail-risky distribution induces lower expected payoffs.

To quantify our comparison and the "sufficiently" high marginal cost of adaptation, we rely on methods similar to those of dynamic programming. For certain classes of distributions, we derive a lower bound on payoff gains that result from communication (Proposition 4). We use this bound to link tail risk to the choice between communication and delegation. For the two-sided generalized Pareto distribution, we characterize the locus of indifference between the two institutions (Proposition 5), and show that in more

[4] We borrow the term tail risk from the finance literature. Mandelbrot (1963) initiated the study of return distributions with fatter tails than the Gaussian distribution. The term is now also used for distributions that have heavier tails than the exponential distribution. Since we use the same stochastic orders whether the distributions have finite or infinite supports, we use the term "tail risk" throughout.

[5] Since scaling is possible for any distribution, it is without loss of generality to consider transformations that keep the variance constant.

tail-risky environments there is more delegation at the optimum. Comparing the Gaussian distribution to the more tail-risky Laplace distribution, we confirm that there is more communication in the less tail-risky Gaussian environment (Proposition 6).

Last but not least, we consider environments that feature thin – sub-exponential – tails, versus those that feature heavy – super-exponential – tails. This comparison entails distributions of the absolute values that are ordered in both the convex transform order and the uniform conditional variability order. Consequently, our result, that communication payoffs decrease as tail risk increases, extends to these environments (Proposition 7).

The remainder of the paper is organized as follows. We first revisit the *Deepwater Horizon* example in more detail. After discussing the related literature, we present our formal model in Section 2. Equilibria of the communication game are derived in Section 3. This section also studies the impact of linear transformations on equilibria and payoffs. We use the uniform and triangular distributions to illustrate our results. In Section 4, we consider increasing, convex transformations of the state variable. We combine stochastic orders of state distributions to compare equilibria and payoffs for different classes of distributions. In Section 5, we introduce a dynamic programming method to quantify gains from communication. We apply our findings to the generalized Pareto distribution and the Gaussian distribution. An extension to a comparison of thin versus heavier tailed distributions is given in Section 6. In Section 7, we conclude. All proofs are in the appendix.

Illustrative example

In the following well-documented example, decision-making based on communication, in an environment with an increased risk of extreme events, resulted in a catastrophe. We do not claim that our model is an adequate description of the case; we think, however, that it illustrates some of the fundamental forces at work. The information provided here is based on the report of the National Comission on the BP Deepwater Horizon oil spill and offshore drilling (2011); on pages viii-ix the report says:

> "[...] the oil and gas industry began to move offshore. The industry first moved into shallow water and eventually into deepwater, where technological advances have opened up vast new reserves of oil and gas in remote areas—in recent decades, much deeper under the water's surface and farther offshore than ever before. The *Deepwater Horizon* was drilling the Macondo well under 5,000 feet of Gulf water, and then over 13,000 feet under the sea floor to the hydrocarbon reservoir below. It is a complex, even dazzling, enterprise. The remarkable advances that have propelled the move to deepwater drilling merit comparison with exploring outer space. The Commission is respectful and admiring of the industry's technological capability.
>
> But drilling in deepwater brings new risks, not yet completely addressed by the reviews of where it is safe to drill, what could go wrong, and how to respond if something does go awry. The drilling rigs themselves bristle with potentially dangerous machinery. The deepwater environment is cold, dark, distant, and under high pressures—and the oil and gas reservoirs, when found, exist at even higher pressures (thousands of pounds per square inch), compounding the risks if a well gets out of control. The *Deepwater Horizon* and Macondo well vividly illustrated all of those very real risks. When a failure happens at such depths, regaining control is a formidable engineering challenge—and the costs of failure, we now know, can be catastrophically high."

Between 1990 and 2009, the oil production industry in the Gulf of Mexico made drastic moves in their drilling locations, from shallow to deepwater wells. This change in production – measured by an increase in oil from deepwater wells from 4% to 80% of the total volume (p.73) – was met by varied conditions for drilling at great depths. We think that an increase in tail risk (that we model by an increasing, convex transformation of the distribution) captures these changes in the drilling conditions quite well.

The rig *Deepwater Horizon* was drilling the *Macondo* well in the Gulf of Mexico, when in 2010 a blowout with catastrophic consequences occurred. *BP*, the owner of the drilling rights (the receiver), relied on subcontractors (the sender), to perform the drilling. *BP* had budgeted money and time (p.2), which we model as the "status quo" that is optimal if the expected conditions are realized. Every adaptation away from the planned procedures was costly to *BP*, with costs increasing in the length of the resulting delay. Hence *BP* responded conservatively to proposed changes (p.125). For example, *BP* decided to continue drilling with unaltered procedures, despite the fact that experts suggested significant changes (*BP* relied on 6 instead of 16 centralizers (p.97), changed the cementing process (p.100), etc.). Our modeling of increasing disagreement, that results in a receiver response with slope less than one, captures this feature. The state in our model may be seen as the specific drilling conditions, for example the pressure conditions below the seabed. Pressure can be unexpectedly high or low. One stylized way to capture these ideas is to assume a symmetric distribution of the state.

The report (p.122) states that "Most, if not all, of the failures at *Macondo* can be traced back to underlying failures of management and communication." Our model indicates that for such a change in the drilling environment, in which extreme realizations become more likely, delegation to the expert outperforms relying on communication.

Related literature

Ours is a contribution to the literature on adaptation in organizations. Alonso et al. (2008) and Rantakari (2008) investigate whether decision-authority should reside at the top of a hierarchy or further down at the level of division-management. These papers, as ours, use the communication model with linear state-dependent bias that was first studied by Melumad and Shibano (1991). Imperfect profit sharing in their models, and adaptation costs in ours, provide a micro foundation for such linear conflicts. Since the adaptation costs in our model are increasing in the size of the adjustment, the wedge between the expert's and the receiver's objective is largest at the extremes of the support. This gives a natural connection to catastrophic outcomes in extreme states, and to such outcomes becoming more likely if the state distribution features heavier tails. Our analysis can be applied directly to situations in which the state can *a priori* only take positive values. Moreover, it can be extended to the Crawford and Sobel (1982) model with disagreement everywhere. We leave this to future work.

Other recent contributions to the adaptation literature include Rantakari (2013), Dessein et al. (2022), and Liu and Migrow (2022). Rantakari (2013) allows firms to choose the compensation and the authority structure jointly. He finds that firms that operate in volatile environments are characterized by decentralized decision making and a compensation with focus on performance at the division level. Dessein et al. (2022) provide a theoretical model predicting that an environment that is more volatile locally results in more decentralized decision making only when the need for coordination across sub-units is low. Liu and Migrow (2022) analyze a model of disclosure with information acquisition. They show that the distribution of an uncertainty parameter has an important impact on the optimal allocation of decision-rights in their problem.

We bring new tools to this literature which typically focuses on volatility in the sense of an increase in the variance of a uniform state. Because we study the impact of heavier tails on unbounded supports, instead of the usually assumed compact state space, we need to build our model from scratch. We prove existence and uniqueness of equilibria, and study the role of risk induced by linear and increasing, convex transformations. We provide comparative statics in terms of stochastic orders which have not been studied before in the context of strategic communication.

Related cheap talk models with endogenous conflicts are Deimen and Szalay (2019) and Antić and Persico (2020). Antić and Persico (2020) consider various ways in which conflicts can arise endogenously, e.g. trading in a stock market prior to communication in a firm. In Deimen and Szalay (2019) a sender acquires noisy signals about a multidimensional state. Depending on the sender's choice of information, conflicts with the receiver arise. That paper shows that communication is better than delegation in a multidimensional elliptical two-sided generalized Pareto environment with heavy tails. Our analysis here builds on our earlier work and provides extensions and generalizations in various directions.

The main new perspective that we bring to the comparison of communication and delegation is the impact of arbitrarily large conflicts. This complements the focus of Dessein (2002), who is the first to study this comparison in the seminal paper of Crawford and Sobel (1982). He shows that whenever interests are sufficiently aligned such that influential communication is possible, the receiver prefers to delegate. As the conflict between sender and receiver becomes arbitrarily small, Dessein (2002) shows that payoffs from unconstrained delegation approach first-best payoffs faster than those arising from strategic communication.[6] In our setup, increasing tail risk has such a detrimental impact on communication that delegation becomes relatively better. Compared to the case of a constant, additive bias, our linear, increasing bias makes it even more surprising that the receiver prefers to delegate when extreme states and extreme disagreement become more likely.

Chen and Gordon (2015) also perform distributional comparative statics in strategic information transmission. They show that information transmission is improved when ideal choices are closer ("games are nested") and a regularity condition is satisfied. As an application, they compare communication to delegation for a Beta distribution. They state conditions on bias and variance such that informative communication is feasible and dominates delegation. Key differences between the papers are that we allow for an agreement point, for unbounded supports, and that we consider increasing, convex transformations.

With few exceptions, the economic theory literature has paid little attention to the shape of distributions. Jewitt (2004) offers an insightful overview of problems in which shape matters. He provides connections among partial orders that describe shape, among them van Zwet's convex transform order. More recently, Di Tillio et al. (2021) show that shapes of distributions, measured by the convex transform order, have a decisive effect on the amount of information contained in a given number of highest sample realizations, compared to the same number of randomly selected data. For example, a given number of highest bids in an auction can contain more or less information than the same number of randomly chosen bids. We study the impact of shape of distributions in cheap talk games. We find that increasing, convex transformations decrease the gains under communication. These transformations do not impact the delegation payoff, more often rendering delegation optimal to communication.

2. Model

We consider a game with two players, a sender S and a receiver R. Sender and receiver have preferences that reflect a common adaptation motive captured by quadratic payoffs that depend on an action $y \in \mathbb{R}$, and on the realization θ of the state of the world Θ. For the sender,

$$u_S(y,\theta) = -(y-\theta)^2 .$$

The receiver faces an additional cost of adaptation $c(y) = \gamma \cdot y^2$, with $\gamma > 0$ such that $u_R(y,\theta,\gamma) = -(y-\theta)^2 - \gamma \cdot y^2$. Defining $a := \frac{1}{1+\gamma}$, the ideal choice functions of sender and receiver are $y_S(\theta) = \theta$ and $y_R(\theta) = a \cdot \theta$, respectively, where $a \in (0,1)$ as $\gamma > 0$. Because of the additional cost, the receiver adapts more conservatively than the sender. The parameter a measures the *alignment of interests*,

[6] See Dilmé (2022) for approximate characterizations of strategic information transmission equilibria with small biases.

with higher values corresponding to more alignment.[7] Since positive affine transformations of utility functions describe the same preferences, we conveniently merge the receiver's motives into one loss function and write[8]

$$u_R(y, \theta, a) = -(y - a \cdot \theta)^2 .$$

The state of the world Θ is a random variable with a common prior distribution F with density f. The support is either the bounded interval $\mathcal{S} = \left[-\overline{S}, \overline{S}\right]$, or unbounded, $\mathcal{S} = \mathbb{R}$. We assume that the density is logconcave, implying a finite variance σ^2, and symmetric, that is $f(x) = f(-x)$, together implying a zero mean.[9]

The sender privately learns the realization of the state, θ. The receiver can choose to communicate with the sender (*communication*). In this case, a sender strategy maps states into distributions over messages, $M_S : \mathcal{S} \to \Delta M$, and a receiver strategy maps messages into actions $Y_R : M \to \mathbb{R}$, where M has the cardinality of the continuum. Strict concavity of payoffs implies that a restriction to pure receiver strategies is without loss of generality. As a simple alternative, the receiver can choose to delegate decision-making to the sender (*delegation*), in which case a sender strategy maps states into actions, $Y_S : \mathcal{S} \to \mathbb{R}$.[10] We solve for Bayes Nash equilibria ("equilibria") of the game.

3. Equilibria and payoffs

3.1. Equilibria of the communication game

Equilibria in our cheap talk game have the typical partitional structure, as the payoffs satisfy the single crossing condition. A partitional equilibrium is characterized by a sequence of *critical types*, $\boldsymbol{t}^n = \left(t_i^n\right)$, with $t_{i-1}^n < t_i^n$ and index n, relating to the number of induced actions. Sender types within an interval, $\left(t_{i-1}^n, t_i^n\right)$, induce the same action; critical types, t_i^n, are indifferent between inducing the action in the interval below or the action in the interval above. As we show in Proposition 1 below, for any finite number of induced actions equilibria are symmetric in our model. For notational simplicity we, therefore, take $t_i^n \geq 0$ and denote the critical types below zero by $-t_i^n$ for all i and n. Receiving a message that indicates that $\theta \in [\underline{t}, \bar{t})$, the receiver updates her belief by forming the conditional expectation $\mu(\underline{t}, \bar{t}) = \mathbb{E}\left[\Theta | \Theta \in [\underline{t}, \bar{t})\right]$. For equilibrium critical types $\boldsymbol{t}^n$, we define

$$\mu_i^n := \mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}^n, t_i^n\right)\right] \text{ for } i = 1, \ldots, n \text{ and } \mu_{n+1}^n := \mathbb{E}\left[\Theta | \Theta \geq t_n^n\right]. \tag{1}$$

Thus, the receiver's equilibrium action given a message indicating $\theta \in \left[t_{i-1}^n, t_i^n\right)$ is $a \cdot \mu_i^n = \arg\max_y \mathbb{E}[u_R(y, \theta, a) | \theta \in \left[t_{i-1}^n, t_i^n\right)]$. The indifference conditions of critical types that determine partitional equilibria are given by

$$t_i^n - a \cdot \mu_i^n = a \cdot \mu_{i+1}^n - t_i^n, \quad \text{for } i = 1, \ldots, n. \tag{2}$$

Symmetric equilibria belong to one of two classes, depending on whether the total number of induced actions is even or odd. In an *Even equilibrium*, type $\theta = 0$ must be a critical type and the equilibrium can be characterized setting $t_0^n = 0$. In an *Odd equilibrium*, a symmetric interval around zero is part of the equilibrium and we omit t_0^n from the construction.[11] For an illustration with $n = 1$, see Fig. 1. The step function depicts the receiver's actions.

Proposition 1.
i) For all n, there exist an essentially unique Even equilibrium, that is symmetric and induces $2(n+1)$ actions, and an essentially unique Odd equilibrium, that is symmetric and induces $2n + 1$ actions.
ii) Even and Odd equilibrium thresholds and actions converge pointwise for $n \to \infty$. We call the limits limit equilibrium.[12] *In particular, we have $\lim_{n\to\infty} t_1^n = 0$, $\lim_{n\to\infty} t_i^n < \lim_{n\to\infty} t_{i+1}^n$, and $\lim_{n\to\infty} t_n^n < \infty$.*

Part i) of Proposition 1 proves the existence and uniqueness of partitional equilibria for arbitrary finite n. An analogous characterization of partitional equilibria for the special case of the Laplace distribution is given in Deimen and Szalay (2019). Proposition 1

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[7] In terms of the literature, the sender's bias is state dependent and equals $(1 - a) \cdot \theta$.

[8] Note that $u_R(y, \theta, \gamma) = -\frac{1}{a}(y - a \cdot \theta)^2 - (1 - a) \cdot \theta^2$, with $\gamma = \frac{1-a}{a}$. The transformation obviously affects levels of utility, but does not impact choices at any margin. The ideal choice functions of the two specifications are the same, and, moreover, the specifications feature the same tradeoffs when choosing among institutions of decision-making.

[9] Symmetry is not essential for our analysis. In some applications a one-sided version of the model is more suitable; changes are straightforward and left to the reader.

[10] We focus on simple unmediated one-round cheap talk. Alonso and Rantakari (2022) identify conditions under which this achieves the maximum payoff among arbitrary mediation rules. We focus on simple unconstrained delegation as this seems natural given our applications. We are, for example, not aware of any restrictions that experts are committed to when trying to prevent a threatening catastrophe. This rules out more complete contracts that allow, for example, for optimal delegation (Holmström (1977), Alonso and Matouschek (2008)). As a theoretical benchmark, optimal delegation would always outperform communication. The comparison to constrained optimal forms of delegation, such as interval delegation, is an interesting open question that is left for future research.

[11] The construction simplifies notation at the expense of characterizing only equilibria with three or more actions. In addition, there exist a babbling equilibrium inducing only one action and a binary equilibrium inducing two actions where zero is the threshold and the actions correspond to the means truncated to positive and negative realizations, respectively.

[12] We do not rule out the existence of other infinite equilibria.

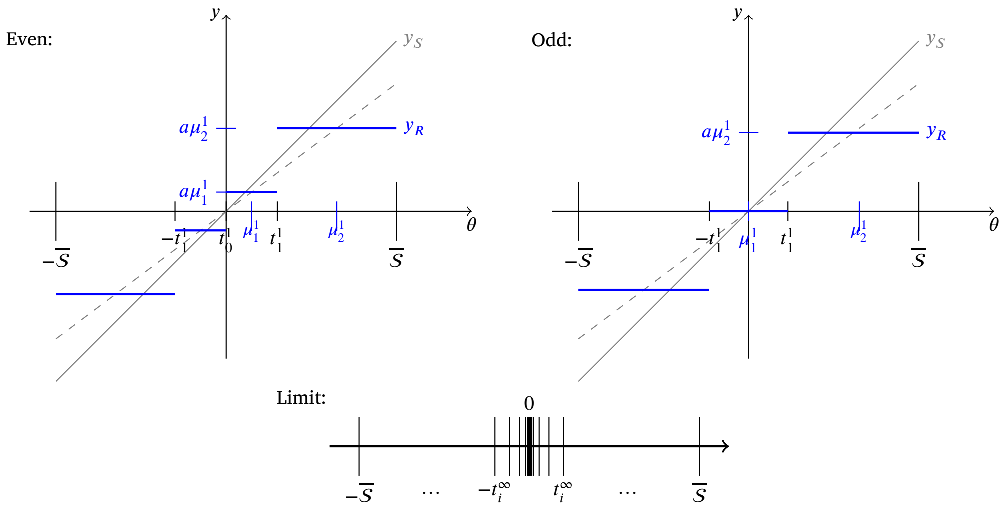

Fig. 1. Partitional equilibria for a uniform distribution. Even and Odd equilibria for $n = 1$ and $a = 0.75$. In a limit equilibrium, intervals around the prior mean 0 get arbitrarily small as $n \to \infty$.

generalizes the result to all symmetric distributions with a logconcave density. Note that the support can be bounded or unbounded. Logconcavity of the distribution and alignment $a \in (0, 1)$ together imply that the solution of a certain forward difference equation is monotonic in the initial value, which we use to prove uniqueness.

Part ii) of the proposition proves that the limit as $n \to \infty$ also is an equilibrium.[13] The partition of a limit equilibrium is illustrated in the bottom panel of Fig. 1. A limit equilibrium has a finite highest critical type, $\lim_{n\to\infty} t_n^n < \infty$, even if the support is unbounded. The reason is that for a distribution with a logconcave density, the mean residual life, $\mathbb{E}[\theta - t_n^n|\theta > t_n^n]$, is decreasing towards zero as $t_n^n \to \infty$. This insight is new to the literature, which typically assumes a compact state space. Equilibrium critical types and actions converge pointwise to limit equilibrium critical types and actions. Intuitively, in a limit equilibrium, starting at the highest critical type, we can calculate the second highest equilibrium action and threshold, proceeding step by step towards zero, which is an accumulation point.

3.2. Communication gains

We first focus on the gains from communication. Define the random variable μ^n of conditional expectations on the discrete support $\left(\pm\mu_i^n\right)_{i=1}^{n+1}$, with μ_i^n (given in equation (1)) derived from the equilibrium partition (t_i^n). As is standard in cheap talk games with quadratic losses, the expected equilibrium utility is a function of the *(ex ante) expected residual variance* after communication, $\mathbb{E}[var^n]$, where var^n is the random variable of conditional variances, conditional on the equilibrium partition. The expected residual variance measures the expected uncertainty that is left after communication has taken place. By the *law of total variance*, the expected residual variance equals the prior variance minus the variance of the inferred posterior means after communication,

$$\mathbb{E}[var^n] = \sigma^2 - var(\mu^n). \tag{3}$$

The variance of the inferred posterior means, $var(\mu^n)$, measures the expected informational *gain from communication*. Communication performs better if the expected residual variance is smaller, or equivalently, if the variance of the inferred posterior means is higher. For our comparative statics analysis, it turns out that the latter object, $var(\mu^n)$, is analytically more convenient to work with.

3.3. Linear transformations

We now begin to investigate the impact of risk on communication equilibria and payoffs. We first study changes in the distribution of the state that are induced by *linear* transformations of the random variable, $\theta \mapsto c \cdot \theta$. Linear transformations that stretch the state space make extreme realizations uniformly more likely, and thus increase the variance. Symmetry of the density implies that we can write $f(\theta) = \kappa \frac{1}{\sigma} \psi\left(\frac{\theta^2}{\sigma^2}\right)$, where κ is a normalizing constant and ψ is a function that captures the shape of the distribution. So,

[13] While the partitional form of equilibria is known from the seminal work of Crawford and Sobel (1982), the structure of the limit equilibrium is closest in spirit to Alonso et al. (2008) and Rantakari (2008). Gordon (2010) offers the first systematic account of the existence of infinite equilibria. We add to this literature by highlighting the role of distributions and, in particular, the role of logconcavity for existence and uniqueness. Logconcavity of the density and a receiver response with a slope less than one – not necessarily constant – provides a micro-foundation for regularity properties that are often imposed in the literature (e.g., *condition M* in Crawford and Sobel (1982) or a *regular receiver response* in Gordon (2010)).

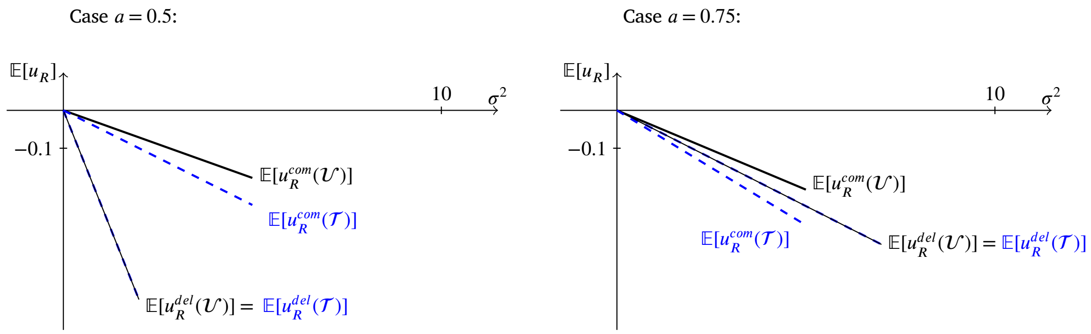

Fig. 2. Payoffs are linear functions of the variance σ^2; communication payoffs (derived in Lemma 2) in a limit equilibrium for uniform $\mathcal{U}$ (black, solid) and triangular $\mathcal{T}$ (blue, dashed) distributions, and delegation payoff (dashed-solid), for $a = 0.5$ (left panel) and $a = 0.75$ (right panel). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

keeping the shape of the distribution fixed, we can scale the distribution by changing the standard deviation σ. As the next lemma shows, equilibrium strategies are linear in the standard deviation, and utilities are linear in the variance.

Lemma 1. *Fix the shape of the distribution $\psi(\cdot)$.*
i) The receiver's expected utility in any equilibrium of the communication game is a linear function of the variance σ^2:

$$\mathbb{E}[u_R^{com}\left(y_R, \Theta, a, n\right)] = -a^2\left(\sigma^2 - var(\mu^n)\right) = -a^2\left(1 - \ell(a, n, \psi)\right)\sigma^2,$$

for some function $\ell(a, n, \psi)$ that is independent of σ^2.
ii) The receiver's expected utility under delegation is $\mathbb{E}[u_R^{del}\left(y_S, \Theta, a\right)] = -(1-a)^2\sigma^2$.

The first statement follows from a change of variables (a linear transformation) to *standardized* random variables. We use the law of total variance to write the receiver's equilibrium expected utility as a function of $\sigma^2 - var(\mu^n)$ in place of the expected residual variance (as explained above). We show that the conditional means are linear in the standard deviation, and thus the gain from communication $var(\mu^n)$ is linear in the variance.

The second statement considers unconstrained delegation to the sender, as a simple alternative to communication. Under delegation, there is no loss of information, as the informed sender takes the action $y_S = \theta$. Sender and receiver, however, disagree on the optimal action by $(1-a)\theta$. Note that communication dominates delegation for $a \leq \frac{1}{2}$. The reason is that even a babbling equilibrium, which is the worst for everyone among all equilibria of the communication game, results in a payoff of $-a^2\sigma^2$.

Expected utilities, whether arising from communication or from delegation, are linearly decreasing in the variance. A higher variance thus results in lower expected utilities under both institutions. However, by linearity, a higher variance never results in a change of the optimal choice of institution in our model, all else equal.

Corollary 1. *Fix the shape of the distribution $\psi(\cdot)$. The choice between delegation and communication in any equilibrium of the communication game, for any fixed number of induced actions, is independent of the variance σ^2.*

This is a direct consequence of σ being a scale variable.[14] By implication, if one mode of decision-making is better than the other for some level of variance, then it is better for every level of variance. In other words, when comparing the optimal choice of institution for two distributions with different shape, it is without loss of generality to scale the distributions such that they have the same variance. Recall that, by Lemma 1, the delegation payoff is only a function of the variance but not of the shape of the distribution.

In terms of the oil-production example, one could think of a linear transformation of the state as of a replication of similar activities, such as adding an on-shore well, where the drilling conditions do not change much. Our result states, that this kind of change in the production does not require a change of the decision making procedure.

Fig. 2 illustrates Corollary 1. Both panels together show that the precise ranking of the payoffs depends on the shape of the distribution as well as the alignment of interest, but not on the level of the variance: for $a = 0.5$, communication is better than delegation (left panel); for $a = 0.75$, delegation is better (worse) than communication under a triangular (uniform) distribution (right panel) for *all* σ^2. To better understand the payoff difference between the uniform and triangular distributions, we now investigate the impact of the shape of the distribution on the communication payoff. We will then come back to the comparison to the delegation payoff.

[14] Equivalent observations have been made in the literature in models in which the state follows a uniform distribution (Alonso et al. (2008), Rantakari (2008)). We extend the result to all scalable distributions. The shape of the distribution does not matter, as long as it is fixed.

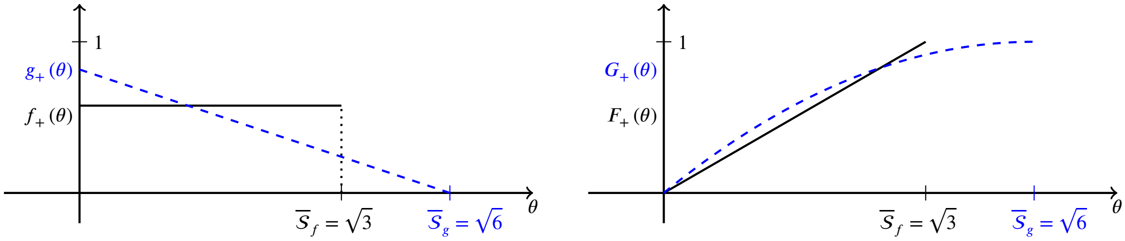

Fig. 3. Uniform distribution F_+ (solid black) and Triangular distribution G_+ (dashed blue) both with variance $\sigma^2 = 1$.

4. Increasing, convex transformations

Given that linear transformations are neutral with respect to the communication/delegation choice, it is natural to conjecture that convex transformations have an effect. Therefore, we now study changes in symmetric distributions of the state that are induced by increasing, convex transformations of the absolute value of the random variable, $\theta \mapsto T(\theta)$ for $\theta \geq 0$, and their impact on communication equilibria and payoffs. Note that the composition of an increasing, convex and a linear transformation remains increasing and convex. Thus by Corollary 1, and without loss of generality, we will focus on increasing, convex transformations $T(\theta)$ that keep the variance constant. Geometrically, this means that $T(\theta)$ must cross the 45° line exactly once from below. In words, the transformation decreases small realizations and increases large ones. This then implies that the cdf of θ crosses the cdf of $T(\theta)$ once from below; recall that both are defined on the positive part of the support. The transformed cdf thus has more mass in the tail of the distribution. In other words, increasing, convex transformations make extreme realizations of the state disproportionally more likely. They are hence a natural way to think of a disproportional increase in risk, which we call *tail risk*.

We will introduce two stochastic orders as our measure of this tail-riskiness. First, two distributions are ordered in the *convex transform order* if one is a convex transformation of the other. In terms of equilibrium analysis, we will show that the convex transform order implies an ordering of the quantiles at the equilibrium thresholds (Proposition 2). In other words, convex transformations imply monotonic shifts in the quantiles at the equilibrium thresholds. Thus, relatively more extreme actions become more likely. In order to obtain a clear ranking of the equilibrium payoffs, more structure is needed. We introduce a second stochastic order, the *uniform conditional variability order*, which gives an additional measure for increased tail-riskiness that applies to any given truncation. Both orders together imply that the difference of the densities has exactly two sign changes on the positive part of the support. We will show that increasing tail-risk in this combined sense implies a monotone comparison of the equilibrium payoffs (Proposition 3).

Intuitively, more mass in the tail implies that extreme events are more likely. Combined with extreme conflicts in the tail, this implies an increased potential for catastrophes. In the oil-drilling example, one could think of a convex transformation of the state as of a change from on-shore to deep-water oil-production, where drilling conditions are relatively more likely to be extreme, e.g., they feature very 'different geological pressures and formations' (National Comission on the BP Deepwater Horizon oil spill and offshore drilling (2011) p.299).

4.1. Equilibrium quantiles

Consider two distinct random variables Θ_f and Θ_g with distributions F and G, and symmetric densities f and g, respectively. Let $\Theta_{f_+} := |\Theta_f|$ and $\Theta_{g_+} := |\Theta_g|$ denote the absolute values of these random variables (or equivalently, by symmetry, the random variables with distributions truncated to the positive halves of their supports). The respective densities of the cdfs F_+ and G_+ are $f_+(x) = 2f(x)$ and $g_+(x) = 2g(x)$ for $x \geq 0$. By symmetry, it is without loss of generality and analytically convenient to study the one-sided distributions F_+ and G_+. The economic intuition, however, is easier to convey by means of the two-sided distributions F and G. In what follows, we therefore go back and forth between the two representations.

We illustrate our assumptions and results graphically with two prominent distributions: the *uniform distribution* represents F and the *triangular distribution* represents G. See Fig. 3 for an illustration of $f_+(\theta) = \frac{1}{\sqrt{3}}$, $F_+(\theta) = \frac{\theta}{\sqrt{3}}$, $g_+(\theta) = \frac{\sqrt{2}}{\sqrt{3}}\left(1 - \frac{\theta}{\sqrt{6}}\right)$, and $G_+(\theta) = 1 - \frac{(\sqrt{6}-\theta)^2}{6}$. The right panel shows that the cdfs cross once; for future reference, note that the difference of the densities $g_+ - f_+$ in the left panel has two sign changes, from positive to negative to positive. Moreover, as discussed above, the assumptions of convexity and equal variances imply that $\mathcal{S}_f \subset \mathcal{S}_g$ for these two distributions, and indeed, for any two distributions with *bounded* supports. More generally, we always have $\mathcal{S}_f \subseteq \mathcal{S}_g$.

To abbreviate notation, we will write $G^{-1}F$ for $G^{-1} \circ F$. Note that we can think of Θ_{g_+} as generated from Θ_{f_+} by the increasing transformation $T\left(\theta_{f_+}\right) = G_+^{-1}F_+\left(\theta_{f_+}\right)$. The distributions are thus related by the condition $G_+\left(T\left(\theta_{f_+}\right)\right) = F_+\left(\theta_{f_+}\right)$. We assume that this transformation is increasing, convex and twice continuously differentiable. For an illustration of this condition for the uniform F_+ and triangular G_+ distributions with $G_+^{-1}F_+(\theta) = \sqrt{6}(1 - \sqrt{1 - \frac{\theta}{\sqrt{3}}})$, see the central panel in Fig. 5. Formally, we assume that the distributions are ordered in the *convex transform order*:

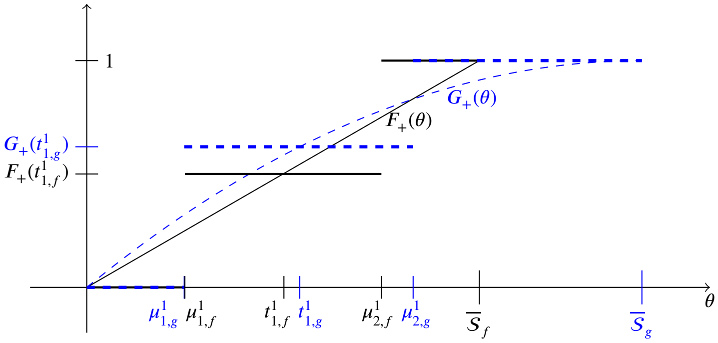

Fig. 4. The discrete distribution of receiver actions in an Even equilibrium for $n = 1$ and $a = 1$, illustrating Proposition 2 for uniform distribution F_+ (black dotted) and triangular distribution G_+ (blue dashed).

Definition 1 *(CTO+). (van Zwet (1964))* Θ_{f_+} is smaller than Θ_{g_+} in the *convex transform order*, that is, $\Theta_{f_+} \leq_c \Theta_{g_+}$, if $G_+^{-1}F_+(\theta)$ is convex in θ on the support of F_+.

The convex transform order is a skewness order: $\Theta_{f_+} \leq_c \Theta_{g_+}$ implies that distribution G_+ is more skewed towards high realizations than F_+. As explained above heuristically, because of the common origin at $\theta = 0$, convexity of $G_+^{-1}F_+(\cdot)$, and $\overline{S}_f \leq \overline{S}_g$, there exists $\hat{\theta}$ such that $G_+^{-1}F_+(\hat{\theta}) = \hat{\theta}$. Therefore, $G_+(\theta) > F_+(\theta)$ for $\theta \in (0, \hat{\theta})$, and $G_+(\theta) < F_+(\theta)$ for $\theta \in (\hat{\theta}, \overline{S}_g)$. This implies that $\frac{G_+(\theta)}{G_+(\hat{\theta})} > \frac{F_+(\theta)}{F_+(\hat{\theta})}$ for $\theta \in (0, \hat{\theta})$, and $\frac{G_+(\theta)-G_+(\hat{\theta})}{1-G_+(\hat{\theta})} < \frac{F_+(\theta)-F_+(\hat{\theta})}{1-F_+(\hat{\theta})}$ for $\theta \in (\hat{\theta}, \overline{S}_g)$. The truncated distributions are thus ordered by first order stochastic dominance: $\frac{F_+(\theta)}{F_+(\hat{\theta})}$ dominates $\frac{G_+(\theta)}{G_+(\hat{\theta})}$ below $\hat{\theta}$, and $\frac{G_+(\theta)-G_+(\hat{\theta})}{1-G_+(\hat{\theta})}$ dominates $\frac{F_+(\theta)-F_+(\hat{\theta})}{1-F_+(\hat{\theta})}$ above $\hat{\theta}$.[15]

Notice that the convex transform order is transitive: For three distributions with cdfs H_+, G_+, and F_+, if $G_+^{-1}F_+$ is increasing and convex and $H_+^{-1}G_+$ is increasing and convex, then $H_+^{-1}F_+ = H_+^{-1}G_+G_+^{-1}F_+$ as the composition of two increasing, convex functions is increasing and convex as well. Moreover, consistent with convex transformations remaining convex under linear transformations, the convex transform order is independent of location and scale (van Zwet (1964)). Formally, two distributions are equivalent in the convex transform order if and only if one random variable is an increasing affine transformation of the other.

Consider now the two-sided distributions F and G. The transformation of the random variable $T(\theta)$ in the two-sided case is concave-convex: concave for $\theta \leq 0$ and convex for $\theta \geq 0$. By symmetry, G has more mass in the tails than F. For this symmetric two-sided case, van Zwet provides an equivalent stochastic (kurtosis) order. Formally, the one-sided distributions satisfy $\Theta_{f+} \leq_c \Theta_{g+}$ if and only if Θ_f is smaller than Θ_g in van Zwet's s-order, $\Theta_f \leq_s \Theta_g$ (van Zwet (1964)). As the s-order implies a higher kurtosis it provides a natural measure of risk.

The s-order and thus the convex transform order have the following implication for equilibria of the communication games.

Proposition 2. *Suppose* $\Theta_{f+} \leq_c \Theta_{g+}$. *Then the quantiles at the equilibrium thresholds for the respective distributions,* $t^n_{i,f}, t^n_{i,g}$*, satisfy* $F_+\left(t^n_{i,f}\right) \leq G_+\left(t^n_{i,g}\right)$ *for all n.*

The proposition shows an ordering of the quantiles at the equilibrium thresholds. For an illustration of the Even equilibrium for $n = 1$, see Fig. 4: if $\Theta_{f+} \leq_c \Theta_{g+}$, then all critical types for G_+ are weakly higher than all critical types for F_+ in the quantile space. Note that the ordering only refers to the quantiles at the critical types, but not necessarily to the actions.

In the oil drilling example, the proposition implies that in the more tail-risky deep-water environment, larger adjustments are less likely to be implemented than in the less tail-risky on-shore environment. Note that statement does not refer to the absolute sizes of the different adjustments.

The proof of the proposition uses the convexity of the transformation and Jensen's inequality. We illustrate the most important step of the proof in Fig. 5. The left panel illustrates the Even equilibrium with $n = 1$ and alignment $a = 1$ for the uniform distribution F_+. The equilibrium actions $\mu^1_{1,f}$ and $\mu^1_{2,f}$ are a mean-preserving spread of the equilibrium threshold $t^1_{1,f}$, $t^1_{1,f} = \frac{1}{2}\mu^1_{1,f} + \frac{1}{2}\mu^1_{2,f}$. The central panel illustrates the convex transformation $T = G_+^{-1}F_+$ applied to the equilibrium values under F_+. Due to the convexity of T and Jensen's inequality, the corresponding values on the vertical axis do *not* form a mean-preserving spread, $T(t^1_{1,f}) < \frac{1}{2}T(\mu^1_{1,f}) + \frac{1}{2}T(\mu^1_{2,f})$ (see the red dashed lines). As a consequnce, the quantiles cannot be part of an equilibrium under G. After transforming back to the state space, we show that the critical types need to increase relative to the situation with the same quantiles. This last argument

[15] Equivalent representations of the convex transform order are given in van Zwet (1964) Lemma 4.1.3, for example, that $\frac{F^{-1''}(q)}{F^{-1'}(q)} \leq \frac{G^{-1''}(q)}{G^{-1'}(q)}$ for $0 < q < 1$, or that the density quantile ratio $\frac{G^{-1'}(q)}{F^{-1'}(q)} = \frac{f_+(F_+^{-1}(q))}{g_+(G_+^{-1}(q))}$ is non-decreasing for $0 < q < 1$.

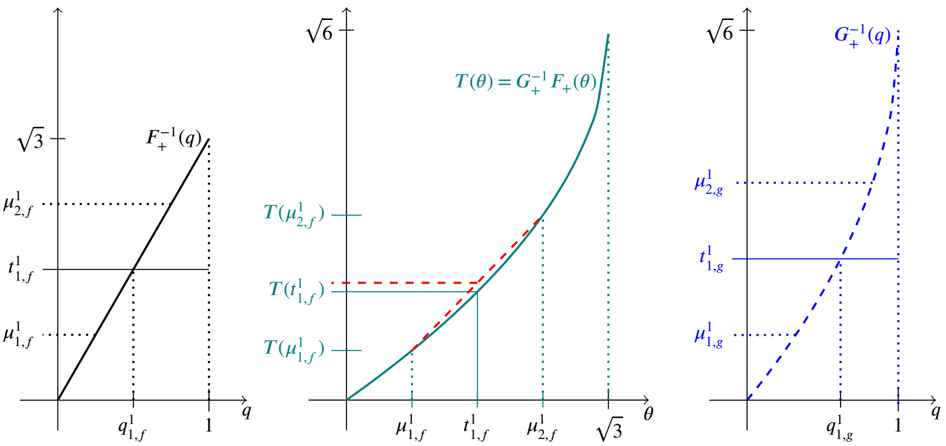

Fig. 5. Even equilibrium for uniform distribution (left panel) and triangular distribution (right panel) in the quantile space both for $n = 1$ and $a = 1$. Convex transformation $T(\theta) = G_+^{-1} F_+(\theta) = \sqrt{6}(1 - \sqrt{1 - \frac{\theta}{\sqrt{3}}})$ (central panel).

uses that the densities are logconcave, which implies that truncated means "move relatively slowly" in their bounds. The right panel shows the equilibrium and the resulting mean-preserving spread for the triangular distribution G_+, $t^1_{1,g} = \frac{1}{2}\mu^1_{1,g} + \frac{1}{2}\mu^1_{2,g}$. The main argument solely relies on convexity. We show in the appendix that the proof extends to $a < 1$ and to an arbitrary number of critical types.

While the proposition allows us to order the quantiles of the equilibrium thresholds, it does not necessarily imply an ordering of the equilibrium payoffs. A sufficient (but not necessary) condition for the ordering of the payoffs is an ordering of all equilibrium actions, $\mu^n_{i,g} < \mu^n_{i,f}$ for all $i \leq n$. If all actions are ordered, then the distribution of the actions for F_+ first order stochastically dominates the distribution of the actions for G_+. This implies that the distribution of the equilibrium actions for F is a mean-preserving spread of the distribution of the equilibrium actions for G.[16] The mean-preserving spread then implies an ordering of the variances, $var(\mu^n_f) > var(\mu^n_g)$, and hence of the payoffs.

Without further assumptions, however, not all the equilibrium actions are always ordered, as our example in Fig. 4 with $\mu^1_{2,g} > \mu^1_{2,f}$ for $a = 1$ shows. In the next subsection, we will address the ranking of the equilibrium actions (1) by imposing more structure on the distributions that we compare, in particular on their likelihood ratio, and (2) by requiring a sufficiently low level of agreement.

4.2. Equilibrium gains

We now focus on the densities of the distributions. The convex transform order implies that $g_+ - f_+$ is positive for small and for large values of θ, so that $g_+ - f_+$ must have *at least* two sign changes. Whitt (1985) introduces a concept of 'increased riskiness' that requires that $g_+ - c \cdot f_+$ has *at most* two sign changes for arbitrary $c > 0$ (and if there are two, they are from positive to negative to positive). The factor c takes care of the relative measure on arbitrary subsets, thus preserving the order under arbitrary truncations. Given the partitional equilibrium structure, the order seems to be tailored to the problem at hand. Whitt shows that this requirement is equivalent to Θ_{f_+} being *uniformly less variable* than random variable Θ_{g_+}:

Definition 2 *(UCV+). (Whitt (1985))* Θ_{f_+} is smaller than Θ_{g_+} in the *uniform conditional variability order*, $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$, if the support of Θ_{f_+} is a subset of the support of Θ_{g_+}, $\mathcal{S}_{f_+} \subseteq \mathcal{S}_{g_+}$, and the ratio $\frac{f_+(\theta)}{g_+(\theta)}$ is unimodal over $\mathcal{S}_{g_+}$, where the mode is a supremum, but Θ_{f_+} is not first order stochastically higher than Θ_{g_+}.

For an illustration of different likelihood ratios on the positive half of the support, see Fig. 6. The ratios are unimodal with interior mode m. As we show in the proof of Proposition 3 below, $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$ implies (again) that there exists some $\tilde{\theta}$ such that $1 - G_+(\theta) < 1 - F_+(\theta)$ for $\theta \in (0, \tilde{\theta})$ and $1 - G_+(\theta) > 1 - F_+(\theta)$ for $\theta \in (\tilde{\theta}, \overline{S}_g)$. Thus, $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$ like $\Theta_{f+} \leq_c \Theta_{g+}$ implies that the distribution of Θ_g has more mass in the tails than the distribution of Θ_f.

The uniform conditional variability order in combination with the convex transform order implies that the difference of the densities has *exactly* two sign changes: $g_+ - f_+$ is positive for small and for large values of θ and negative for intermediate values of θ. We use the unimodal likelihood ratio to order the equilibrium actions and thereby the equilibrium payoffs:

[16] For a definition of mean-preserving spreads see, for example, Mas-Colell et al. (1995) Proposition 6.D.2; for an earlier reference see Hardy et al. (1929)).

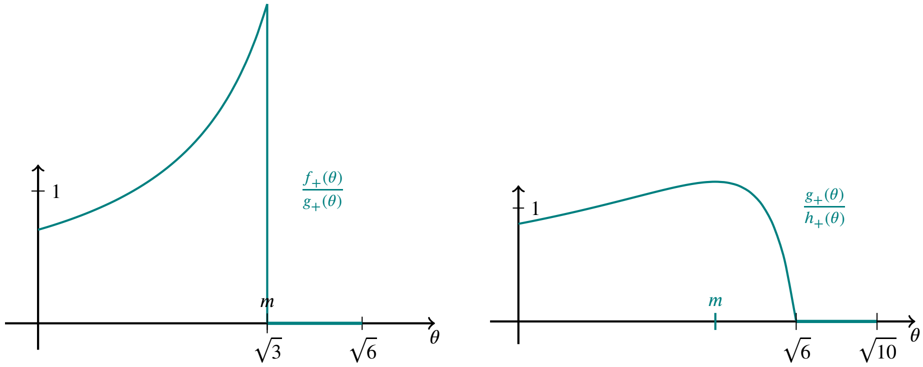

Fig. 6. Uniform conditional variability. Left panel: distributions f_+ (uniform) and g_+ (triangular) with $\frac{f_+(\theta)}{g_+(\theta)} = \frac{1}{\sqrt{2}-\frac{\theta}{\sqrt{3}}}$ satisfy $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$. Right panel: distributions g_+ (triangular) and $h_+(\theta) = \frac{\sqrt{9}}{\sqrt{10}}\left(1-\frac{\theta}{\sqrt{10}}\right)^2$ for $\theta \in \left[0, \sqrt{10}\right]$ with $\frac{g_+(\theta)}{h_+(\theta)} = \frac{\sqrt{20}}{\sqrt{27}} \frac{\left(1-\frac{\theta}{\sqrt{6}}\right)}{\left(1-\frac{\theta}{\sqrt{10}}\right)^2}$ satisfy $\Theta_{g_+} \leq_{uv} \Theta_{h_+}$ (see Lemma A.7).

Proposition 3. *Suppose that F and G have the same variance σ^2 and that $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$ and $\Theta_{f+} \leq_c \Theta_{g+}$. Then, there exists $a' \in (0,1)$, defined in the proof, such that for $a \leq a'$, the distribution of μ_f^n is a mean-preserving spread of the distribution of μ_g^n, implying that*

$$var_f(\mu_f^n) > var_g(\mu_g^n).$$

The proposition implies by equation (3) that the payoffs for distribution f are higher than those for g. Note that this insight extends to distributions f and g with $\sigma_g^2 > \sigma_f^2$ that satisfy the remaining conditions of the proposition, since $-\sigma_f^2 + var_f(\mu_f^n) > -\sigma_g^2 + var_g(\mu_g^n)$. Hence our comparison extends to cases in which one distribution is more risky than the other in the sense of a higher variance *and* a higher kurtosis.

Condition $\Theta_{f+} \leq_c \Theta_{g+}$ implies that for any alignment $a \in (0,1)$ the equilibrium probability distribution of the receiver's actions for F puts more weight on the more extreme actions than that for G. For a relatively small, all thresholds and actions are relatively close to the prior mean. Due to $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$ this implies that the equilibrium actions for f are all farther away from zero than those for g. Taken together, the ranking of quantiles and actions imply that the distribution of μ_f^n is a mean-preserving spread of the distribution of μ_g^n.

Table 1
Even equilibrium values for uniform and triangular distributions for $n = 1$.

Uniform

a	0.25	0.5	0.75	1
$t_{1,f}^1$	0.124	0.289	0.520	0.866
$\mu_{1,f}^1$	0.062	0.144	0.260	0.433
$\mu_{2,f}^1$	0.923	1.010	1.126	1.299
$F_+(t_{1,f}^1)$	0.071	0.167	0.3	0.5
$var(\mu_f^1)$	0.800	0.854	0.908	0.938

Triangular

a	0.25	0.5	0.75	1
$t_{1,g}^1$	0.119	0.287	0.537	0.936
$\mu_{1,g}^1$	0.059	0.141	0.257	0.431
$\mu_{2,g}^1$	0.896	1.008	1.174	1.440
$G_+(t_{1,g}^1)$	0.095	0.221	0.390	0.618
$var(\mu_g^1)$	0.727	0.796	0.867	0.907

Table 1 illustrates our findings. First, the table shows that for sufficiently little alignment a, the receiver's equilibrium actions are closer to the prior mean for G_+ than for F_+. Together with Proposition 2 this implies that the distribution of actions for F is a mean-preserving spread of the distribution of actions for G. Second, the table shows that the ordering of the equilibrium actions is sufficient, but not necessary, for a higher gain from communication: the variance of receiver actions $var(\mu)$ is higher for F than for G for all levels of alignment a in the table.[17]

In terms of the oil example, adapting drilling procedures is very costly, indicating a likely conflict of interest. The proposition states that the gain from communication in a less tail-risky environment, say on-shore drilling, is higher than that in a more tail-risky environment, say deep-water drilling. Thus communicating information in a deep-water environment cannot be expected to work that well, as extreme circumstances, resulting in large disagreement, are disproportionally likely.

5. Delegation versus communication

In this section, we quantify the effects derived in the previous analysis. We provide some interpretation for the term "sufficiently misaligned" interests used in Proposition 3. Moreover, we link our findings back to the comparison of delegation versus communi-

[17] Uniform conditional variability and logconcavity prove useful also outside of communication games; see, for example, Lyu et al. (2023) for an application to information design.

cation. We show that for a given alignment of preferences, a more risky distribution can change the optimal way of decision-making from communication to delegation. All distributions we compare in this section are ranked according to the convex transform order and the uniform conditional variability order.

To quantify the effects and to derive a formula for the gain from communication, we rely on a "dynamic programming" procedure as our technical tool. The slope of the *tail-truncated expectation* function $\phi(t) := \mathbb{E}[\Theta | \Theta \geq t]$ for $t \geq 0$, is a crucial determinant of this value.[18] The case of a linear tail-truncated expectation – the two-sided generalized Pareto distribution – is particularly structured; we treat this in the next subsection. A second class of interest is the class of convex tail-truncated expectation functions. The Gaussian distribution is a prominent case with this property, which we treat thereafter.

The gain from communication in a limit equilibrium can be quantified as follows.

Proposition 4. *Suppose that $\phi(t) := \mathbb{E}[\Theta | \Theta \geq t]$ is convex in $t \geq 0$. Then the variance of μ^n in a limit equilibrium satisfies*

$$var(\mu^\infty) \geq \frac{2}{2 - a \cdot \phi'(0)} \cdot \phi(0)^2.$$

If $\phi(t)$ is linear in $t \geq 0$, then the condition is satisfied with equality.

The (lower bound on the) variance of μ^∞ is a product of two terms. The factor $\phi(0)^2 = \mathbb{E}[\Theta | \Theta \geq 0]^2$ measures the amount of information that is transmitted by binary communication, when dividing the state space into positive and negative realizations. The factor $\frac{2}{2 - a \cdot \phi'(0)}$ captures (a lower bound on) the additional information contained by dividing each half into a countable infinity of subintervals. The latter term depends on the slope of the tail-truncated expectation, $\phi'(t)$. The slope is constant for a linear tail-truncated expectation function which therefore yields a closed form solution. The slope is increasing for convex tail-truncated expectations. We thus obtain a lower bound on the variance of equilibrium actions by using the minimal slope of the conditional expectation, which amounts to the slope at zero, $\phi'(0)$.

5.1. The linear case: generalized Pareto distribution

The tail-truncated expectation $\phi(t) = \mathbb{E}[\Theta | \Theta \geq t]$ is linear in $0 \leq t < \overline{S}$ if and only if the state is distributed according to a *two-sided generalized Pareto distribution*.[19] For this class, the density is

$$f(\theta; \delta, s) = \frac{1}{2s} \left(1 + \delta \frac{|\theta|}{s}\right)^{-\frac{1}{\delta} - 1} \quad \text{for } \theta \in \left[\frac{s}{\delta}, -\frac{s}{\delta}\right], \tag{GP}$$

where $s \in (0, \infty)$ is a scale parameter and $\delta \in [-1, 0)$ is a shape parameter.[20] The variance of the distribution is $\sigma^2(s, \delta) = \frac{2s^2}{(1-\delta)(1-2\delta)}$.

Increases in scale s, for a given shape δ, make the support, $[\frac{s}{\delta}, -\frac{s}{\delta}]$, wider, and move equilibrium actions further away from the mean. Increases in shape δ, for a given support, move equilibrium actions closer to the mean. Any two distributions f, g in this generalized Pareto class with parameters s, δ and s', δ' respectively, such that $0 > \delta' > \delta \geq -1$, $s' < s$, and $-\frac{s}{\delta} < -\frac{s'}{\delta'}$, satisfy the convex transform order $\Theta_{f+} \leq_c \Theta_{g+}$ and the uniform conditional variability order $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$ (see Lemma A.7 in the appendix).

The class nests many well-known distributions: the case $\delta = -1$ is the uniform distribution, $\delta = -\frac{1}{2}$ is the triangular distribution, and the limit case $\delta = 0$ is well defined and corresponds to the Laplace distribution. By Proposition 4, for the generalized Pareto environment, the expected utilities arising from communication in a limit equilibrium can be stated in closed form.

Lemma 2. *For the two-sided generalized Pareto distribution with shape $\delta \in [-1, 0)$ and scale $s^2 = \sigma^2 \frac{(1-\delta)(1-2\delta)}{2}$, we have that $\phi'(0) = \frac{1}{1-\delta}$. Hence, in a limit equilibrium, we have*

[18] The computation is based on a procedure which is akin to dynamic programming (for $a = 1$, i.e., identical sender and receiver preferences, it would be dynamic programming in the literal sense). In particular, we compute the expected squared deviation from $\phi(0)$ conditional on the last interval, then conditional on the last two intervals, and so on, proceeding towards zero. In each step, we can simplify the expression using the arbitrage condition of the threshold types, the law of iterated expectations – which links expectations over subintervals to expectations truncated to the tail of the distribution –, and the special form of tail-truncated expectations. If the tail-truncated expectation function is linear in the truncation point, we can carry an exact functional form backwards towards zero, obtaining a closed form expression in the limit. The procedure was developed in (Deimen and Szalay (2019)). Here, we generalize the procedure to the case of convex tail-truncated expectations. Note that our quantitative assessment of communication gains via "dynamic programming" applies to any distribution that becomes relatively more variable towards the tail of its distribution in the sense of a globally increasing residual coefficient of variation (Gupta and Kirmani (2000)). Gupta and Kirmani (2000) show that the residual coefficient of variation, i.e., the ratio of residual variance and mean residual life squared, increases in the truncation point if $\phi(t)$ is convex in t.

[19] In Deimen and Szalay (2019), we derive distributions with a linear tail-truncated expectation from first principles as the unique solution to a differential equation. In that formulation, the solution involves the variance and the slope of the tail-truncated expectation. Here, we observe that we can re-parametrize these distributions to the general Pareto class with scale s and shape δ.

[20] The distribution is constructed from the well-known one-sided generalized Pareto by reflecting the density at the vertical axis. The location parameter is set to zero, so that the mean is zero. The distribution is defined more generally for shape parameters $\delta \in (-\infty, \infty)$, but we restrict attention to the subset that features logconcave tails. We treat the case $\delta \geq 0$ in Deimen and Szalay (2019); these distributions have logconvex tails and an infinite support. The generalized Pareto distribution was introduced by Pickands (1975) in the context of extreme value theory. See also Arnold (2008).

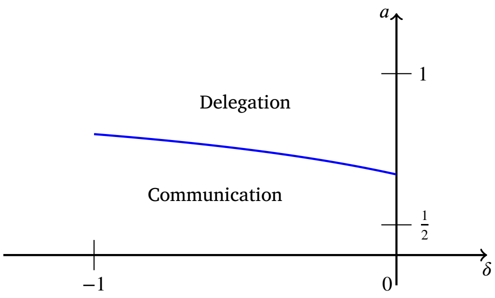

Fig. 7. Delegation versus communication. On the horizontal axis, the shape parameter δ increases from -1 (uniform distribution) to the limit of 0 (Laplace distribution; see Deimen and Szalay (2019)); on the vertical axis, the level of alignment a increases from $\frac{1}{2}$ to 1.

$$var(\mu^{\infty}) = \frac{2 - \frac{1}{1-\delta}}{2 - \frac{a}{1-\delta}}\sigma^2. \tag{4}$$

Equation (4) obtains from applying Proposition 4, noting that $\phi(0) = \frac{s}{1-\delta}$, and using the functional form of the variance. Naturally, $var(\mu^n) \leq var(\mu^{\infty}) \leq var(\Theta)$. For $a \to 0$, the value $var(\mu^{\infty})$ approaches the value of binary communication. Note that for any given alignment $a \in (0,1)$, the value $var(\mu^{\infty})$ is decreasing in δ.[21] A larger shape parameter reduces the value of communication, as less information is transmitted in equilibrium. Thus, within the generalized Pareto class, the shape parameter has a strictly negative impact on the gain from communication for any value of $a < 1$.[22] In the sense of Proposition 3, for this class, the condition of "sufficiently misaligned" interests is always satisfied, i.e., $a' = 1$, and there is no restriction on the alignment parameter.

It is now straightforward to investigate the effect of the shape of the distribution on the optimal choice of institution: communication or delegation.

Proposition 5. *Consider the two-sided generalized Pareto distribution with $\delta \in [-1, 0)$. Suppose the receiver can choose between communication and delegation. Then, delegation is better than communication in any equilibrium of the communication game if $\delta \geq \frac{2-3a}{2-2a}$. Communication in a limit equilibrium is better than delegation if $\delta \leq \frac{2-3a}{2-2a}$.*

While the performance of delegation depends only on the variance of the environment, the performance of communication depends in addition on the shape of the distribution. The fraction of information that is transmitted in a limit equilibrium, $\frac{2-\frac{1}{1-\delta}}{2-\frac{a}{1-\delta}}$, is smaller in environments that feature a larger δ. We depict the comparison in Fig. 7.

Consistent with the literature, delegation dominates communication when interests are relatively well aligned and the receiver is quite responsive to the sender's advice, i.e., $a \geq \frac{2-2\delta}{3-2\delta}$.[23] The comparison in terms of the shape of the distribution adds a new dimension to the literature. For $a \in \left(\frac{2}{3}, \frac{4}{5}\right)$ and a distribution with a low shape parameter, communication is optimal. If the shape parameter is higher, however, delegation is optimal. In other words, an increase of the mass in the tail of the distribution – an increase in tail risk – induces a change in the mode of decision-making from communication to delegation in the named range. Note that the conditions in Proposition 5 are independent of the scale parameter s. This is an illustration of Corollary 1: Proposition 5 compares communication and delegation payoffs for distributions with arbitrary variances.[24]

With regard to the oil example, our analysis suggests that it is not the increase in oil-production but the change in the drilling procedure, say from on-shore to deep-water drilling, that asks for a different decision protocol: delegation instead of communication. Following experts' recommendations, such as installing the adequate gear or waiting for test results, could have mitigated the catastrophe (National Comission on the BP Deepwater Horizon oil spill and offshore drilling (2011) p.125).

[21] For $a = 1$, the value of partitional communication reaches the upper bound of fully revealing communication. For $a = 0$, the receiver's action equals zero for any sender strategy.

[22] Alonso and Rantakari (2022) derive an upper bound on the maximal communication payoff for a set of distributions. In particular, they consider a uniform, a half-triangular, and a truncated exponential distribution. They show that for a constant variance a shift from truncated exponential, to half-triangular, to uniform improves the payoffs under communication. This is in line with our findings. Because they consider a fixed interval support, this shift implies an increase in the expected conflict. Thus in their paper, a shift that reduces the expected conflict can worsen communication. In our paper, better aligned interests imply better communication payoffs.

[23] See, for example, Alonso et al. (2008) and Rantakari (2008) who study a uniform distribution, i.e., $\delta = -1$. See also Dessein (2002).

[24] Chen and Gordon (2015) (Section 5.2) compare delegation and communication for a Beta distribution and an additive bias. They state conditions on the bias and the variance such that informative communication is feasible and dominates delegation.

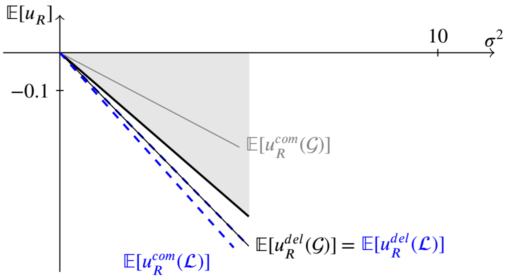

Fig. 8. Communication payoffs in a limit equilibrium for the Laplace distribution $\mathcal{L}$ (blue, dashed) and the lower bound for the Gaussian distribution $\mathcal{G}$ (black, solid) and the delegation payoff (dashed-solid), for $a = 0.68$. The communication payoff in a limit equilibrium for the Gaussian distribution is some ray indicated in gray, in the shaded area.

5.2. A convex case: Gaussian versus Laplace

We here consider two distributions with infinite support. The Laplace distribution features linear tail-truncated expectations; it is the limit case with $\delta = 0$ in the two-sided generalized Pareto class studied in the previous subsection. A leading example within the class of distributions with convex tail-truncated expectations is the Gaussian distribution (see Sampford (1953)).

Lemma 3. *For the Gaussian distribution, $\phi'(0) = \frac{2}{\pi}$. The gain from communication in a limit equilibrium is bounded from below,*

$$var(\mu^\infty) \geq \frac{2}{\pi - a}\sigma^2. \tag{5}$$

The lower bound on the variance of equilibrium actions uses the minimal slope of the truncated expectation, which amounts to the slope at the origin, $\phi'(0) = \frac{2}{\pi}$. Applying Proposition 4 and noting that for the Gaussian distribution $\phi(0)^2 = \frac{2}{\pi}\sigma^2$, one obtains (5).

With this at hand, we can compare delegation to communication for the Gaussian and for the Laplace distribution. Note that the distributions of the absolute values of the Gaussian and the Laplace are ordered as follows: the Gaussian is smaller than the Laplace in the convex transform order; and the Gaussian is uniformly less variable than the Laplace for equal variances (see Lemma A.8 in the appendix). Hence, we can say that the Laplace distribution is more tail-risky than the Gaussian distribution.

Recall that by Corollary 1, the comparison of delegation and communication is independent of the variance. Denote by $\sigma^2_{\mathcal{G}}$ ($\sigma^2_{\mathcal{L}}$) the variance of the Gaussian (Laplace) distribution.

Proposition 6. *i) For any $\sigma^2_{\mathcal{G}}, \sigma^2_{\mathcal{L}}$, if communication is preferred over delegation for the Laplace distribution, then communication is also preferred over delegation for the Gaussian distribution.*

ii) The value of communication in a limit equilibrium is higher for the Gaussian than for the Laplace distribution, if $a < 0.858$ and $\sigma^2_{\mathcal{G}} \leq \sigma^2_{\mathcal{L}}$.

The proposition states that optimally there is more delegation in the more tail-risky environment, for any levels of the variances. Moreover, there is a nonempty set of preference alignment parameters a for which delegation is preferred for the more tail-risky Laplace distribution, whereas communication in a limit equilibrium is preferred for the less tail-risky Gaussian distribution. For an illustration, see Fig. 8.

For part ii), interests are sufficiently misaligned in the sense of Proposition 3, if $a \leq 0.858$. In this case, and for all variances satisfying $\sigma^2_{\mathcal{G}} \leq \sigma^2_{\mathcal{L}}$, the lower bound of the value of communication in a limit equilibrium for the Gaussian distribution outperforms the exact value of communication in a limit equilibrium for the Laplace distribution.

The Gaussian distribution is one example. Similar results can be obtained for any distribution with a logconcave density and a convex tail-truncated expectation function.

6. Thin versus heavy tails

We have so far confined our attention to distributions with a logconcave density – i.e., relatively thin tails. We now expand our analysis to distributions with logconvex, i.e., heavier, tails.

Proposition 7. *Consider two symmetric distributions F, G with support on $\mathbb{R}$ and with the same finite variance σ^2. Suppose that the density f_+ is logconcave and the density g_+ is logconvex. Then, $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$ and $\Theta_{f+} \leq_c \Theta_{g+}$. Moreover, in any informative symmetric equilibrium, there exists $a' \in (0,1)$, such that for $a \leq a'$, the distribution of μ^n_f is a mean preserving spread of the distribution of μ^n_g, implying that*

$$var_f(\mu^n_f) > var_g(\mu^n_g).$$

The payoffs of the communication games for distributions with logconcave tails are thus higher than for those with logconvex tails. Even though Proposition 1 does not apply, there always exists an informative symmetric equilibrium, since binary communication is always feasible.

To prove the proposition, we only need to apply Proposition 3. Hence, we aim at showing that the conditions stated in the proposition imply that the distributions are ranked according to $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$ and $\Theta_{f+} \leq_c \Theta_{g+}$. We first consider the uniform conditional variability order and relate it to the well-known concept of *relative logconcavity*, introduced by Whitt (1985): f_+ is said to be *logconcave relative to* g_+, if $\frac{f_+}{g_+}$ is logconcave. We obtain the following.

Lemma 4. *Consider two symmetric distributions with the same variance and with densities f, g on $\mathbb{R}$ such that $\frac{f_+}{g_+}$ is logconcave, then $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$.*[25]

In particular, note that $\frac{f_+}{g_+}$ is logconcave if f_+ is logconcave and g_+ is logconvex. As a consequence, two distributions satisfying the assumptions in Proposition 7 are ranked according to the uniform conditional variability order, $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$. We note that, in contrast to the uniform conditional variability order, relative logconcavity is a transitive concept.

We next consider the convex transform order, $\Theta_{f+} \leq_c \Theta_{g+}$.

Lemma 5. *Consider two symmetric densities f, g on $\mathbb{R}$. If f_+ is logconcave and g_+ is logconvex, then $\Theta_{f+} \leq_c \Theta_{g+}$.*

Hence the conditions of Proposition 3 are satisfied, and we obtain a mean-and-variance-preserving spread in terms of the underlying state distributions. By the now familiar arguments, this induces a mean-preserving spread in the distributions of receiver actions.

In closing, we note that we have picked the most focal point of comparison: the loglinear (Laplace) distribution which separates logconcave from logconvex distributions. We can pick any other distribution as a point of reference, for example the Gaussian distribution. Distributions that are logconcave relative to the Gaussian distribution are called *strongly logconcave* (Wellner (2013)). In a similar vein, we can consider distributions that are smaller or larger than the Gaussian distribution in the convex transform order. Our insights carry over to these comparisons.

All of our results show that communication tends to perform poorly in environments with heavier tails compared to environments with thinner tails.

7. Conclusion

In this paper, we study the impact of risk on the performance of communication, through transformations of the state variable. In particular, we are interested in the likelihood of extreme events, which in our model is tied to the likelihood of extreme disagreement. We compare the payoffs under communication with those under unconstrained delegation.

We find that linear transformations (changes in the variance for a given shape of the distribution) scale the payoffs under communication as well as under delegation. Increasing, convex transformations (changes in the shape of the distribution that increase the kurtosis for a given variance) only impact the payoffs under communication. We combine the convex transform order with the uniform conditional variability order to rank the payoff gains from communication, assuming adaptation costs of some size for the receiver. Increasing the tail risk through a combination of linear and of increasing, convex transformations more often renders delegation optimal relative to communication.

Finally, we confirm our finding that an increase in tail risk is detrimental for communication when comparing distributions with thin tails to distributions with heavier tails. When extreme events become more likely, communication suffers.

CRediT authorship contribution statement

Inga Deimen: Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Dezső Szalay:** Writing – original draft, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The author declares that she has no relevant or material financial interest that relate to the research described in the paper "Communication in the Shadow of Catastrophe."

[25] From Shaked and Shanthikumar (2007) Theorem 3.A.54, it is known that, for equal supports, relative logconcavity and exactly two sign changes of the difference of the densities imply the uniform variability order. In contrast, we show that the uniform variability order arises from relative logconcavity plus the distributions having the same variance.

Data availability

No data was used for the research described in the article.

Appendix A

Definition A.1. The forward equation is recursively defined as solutions $t_{i+1}(t_{i-1}, t_i)$ to the indifference conditions of types t_i. We denote an arbitrary initial value of t_1 by τ. In particular, for $i = 1$ we have $t_2(0, \tau)$ as solution to

$$2\tau - a\mathbb{E}\left[\Theta | \Theta \in [0, \tau]\right] - a\mathbb{E}\left[\Theta | \Theta \in \left[\tau, t_2(0, \tau)\right]\right] = 0, \tag{6}$$

for $i > 1$ we have $t_{i+1}(t_{i-1}, t_i)$ as solutions to

$$2t_i - a\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] - a\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}(t_{i-1}, t_i)\right]\right] = 0. \tag{7}$$

Lemma A.1. (Szalay (2012)) *(Strict) Logconcavity of the distribution implies that*

$$\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] + \frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] \leq (<) 1.$$

Lemma A.2. *Consider the forward equation. Logconcavity of the distribution and $a < 1$ implies that for all $i = 1, \ldots, n-1$*

$$\frac{dt_{i+1}}{dt_i} = \frac{\left(2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right]\right)}{a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right]} > 1.$$

Proof of Lemma A.2. Consider the forward equation for t_2. The value $t_2(0, \tau)$ is the unique solution to (6). Totally differentiating (6) we find

$$\frac{dt_2}{d\tau} = \frac{\left(2 - a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta | \Theta \in [0, \tau]\right] - a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta | \Theta \in \left[\tau, t_2\right]\right]\right)}{a\frac{\partial}{\partial t_2}\mathbb{E}\left[\Theta | \Theta \in \left[\tau, t_2\right]\right]} > 1,$$

where the inequality follows from Lemma A.1:

$$2 - a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta | \Theta \in [0, \tau]\right] > 1 > a\frac{\partial}{\partial \tau}\mathbb{E}\left[\Theta | \Theta \in \left[\tau, t_2\right]\right] + a\frac{\partial}{\partial t_2}\mathbb{E}\left[\Theta | \Theta \in \left[\tau, t_2\right]\right].$$

Next, consider arbitrary $i = 1, \ldots, n-1$. The sender's solution to the forward equation for t_i is given by (7). Totally differentiating (7) yields

$$\left(2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right]\frac{dt_{i-1}}{dt_i}\right)dt_i$$
$$= a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right]dt_{i+1}.$$

Suppose as an inductive hypothesis that $\frac{dt_i}{dt_{i-1}} > 1$, so $\frac{dt_{i-1}}{dt_i} < 1$. Rearranging, we get

$$\frac{dt_{i+1}}{dt_i} = \frac{\left(2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right]\frac{dt_{i-1}}{dt_i}\right)}{a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right]}$$
$$> 1,$$

which obtains by the inductive hypothesis and Lemma A.1:

$$2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right]\frac{dt_{i-1}}{dt_i}$$
$$> 2 - a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right] - a\frac{\partial}{\partial t_{i-1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_{i-1}, t_i\right]\right]$$
$$> 1 > a\frac{\partial}{\partial t_i}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right] + a\frac{\partial}{\partial t_{i+1}}\mathbb{E}\left[\Theta | \Theta \in \left[t_i, t_{i+1}\right]\right]. \quad \square$$

Lemma A.3. *The last equilibrium threshold t_n^n is bounded from above for all n and $\lim_{n\to\infty} t_n^n < \infty$.*

Proof of Lemma A.3. The statement is trivial for a bounded support.

For an unbounded support, consider the closure condition and define

$$\Delta_n(\tau) \equiv 2t_n(\tau) - a\mathbb{E}\left[\theta \mid \theta \in \left[t_{n-1}(\tau), t_n(\tau)\right]\right] - a\mathbb{E}\left[\theta \mid \theta \geq t_n(\tau)\right].$$

Now, $\Delta_n(\tau) = 0$, for $\tau = t_1^n$. We have

$$\Delta_n\left(t_1^n\right) = 2t_n^n - a\mathbb{E}\left[\theta \mid \theta \in \left[t_{n-1}^n, t_n^n\right]\right] - a\mathbb{E}\left[\theta \mid \theta \geq t_n^n\right] \geq 2\left(t_n^n - a\mathbb{E}\left[\theta \mid \theta \geq t_n^n\right]\right),$$

which follows from $-a\mathbb{E}\left[\theta \mid \theta \in \left[t_{n-1}^n, t_n^n\right]\right] \geq -a\mathbb{E}\left[\theta \mid \theta \geq t_n^n\right]$. For a logconcave distribution, $t - a\mathbb{E}[\theta \mid \theta \geq t]$ is negative for $t = 0$, increasing in t, and goes to ∞ for $t \to \infty$. Therefore, $\lim_{n\to\infty} t_n^n < \infty$ and the sequence t_n^n is bounded above. □

Proof of Proposition 1. The proof is analogous to the proof of Proposition 1 in Deimen and Szalay (2019) which only considers the Laplace distribution, and therefore omitted. Instead of using the functional form of the Laplace distribution one can apply properties of logconcave densities to show the statements. These properties are summarized in Lemma A.1 and Lemma A.2. Moreover, Lemma A.3 shows the existence of a bound. For a detailed version of the proof, we refer the interested reader to the working paper version Deimen and Szalay (2023). □

Proof of Lemma 1. i) Note that any symmetric one-dimensional density is elliptical (Cambanis et al. (1981)). Moreover, elliptical densities can be written as $f(\theta) = \kappa \frac{1}{\sigma}\psi\left(\frac{\theta^2}{\sigma^2}\right)$, where κ is a normalizing constant and ψ is a (density generator) function that captures the shape of the distribution (Gómez et al. (2003)). Thus, the density depends only on the standardized variable $\frac{\theta}{\sigma}$.

We show that equilibrium strategies (t_i^n) and $a \cdot (\mu_i^n)$ are linear in the standard deviation σ, i.e., $\boldsymbol{z}^n = \left(z_i^n\right) = \left(\frac{t_i^n}{\sigma}\right)$ is the sequence of equilibrium critical types for the standardized distribution with unit variance, and $\mathbb{E}\left[\Theta \mid \Theta \in \left[t_{i-1}^n, t_i^n\right]\right] = \sigma\mathbb{E}\left[Z \mid Z \in \left[z_{i-1}^n, z_i^n\right]\right]$, for $Z := \frac{\Theta}{\sigma}$.

Consider a typical equilibrium indifference condition

$$t_i - a\mathbb{E}\left[\Theta \mid \Theta \in \left[t_{i-1}, t_i\right]\right] = a\mathbb{E}\left[\Theta \mid \Theta \in \left[t_i, t_{i+1}\right]\right] - t_i.$$

A change of variables to $z = \frac{\theta}{\sigma}$, and thus $dz = \frac{1}{\sigma}d\theta$, implies that

$$\mathbb{E}\left[\Theta \mid \Theta \in \left[t_{i-1}, t_i\right]\right] = \frac{\int_{t_{i-1}}^{t_i} \theta\kappa\frac{1}{\sigma}\psi\left(\frac{\theta^2}{\sigma^2}\right)d\theta}{\Pr\left(\Theta \in \left[t_{i-1}, t_i\right]\right)} = \frac{\sigma\int_{z_{i-1}}^{z_i} z\kappa\psi\left(z^2\right)dz}{\Pr\left(Z \in \left[z_{i-1}, z_i\right]\right)} = \sigma\mathbb{E}\left[Z \mid \Theta \in \left[z_{i-1}, z_i\right]\right],$$

with $z_i = \frac{t_i}{\sigma}$. Hence the indifference condition can be written as

$$z_i - a\mathbb{E}\left[Z \mid Z \in \left[z_{i-1}, z_i\right]\right] = a\mathbb{E}\left[Z \mid Z \in \left[z_i, z_{i+1}\right]\right] - z_i,$$

which is independent of the variance. As a consequence, the standardized equilibrium thresholds z_i are independent of the variance.

It follows that $var(\mu^n)$ is linear in σ^2, $var(\mu^n) = \ell(n, a, \psi)\sigma^2$, where $\ell(n, a, \psi)$ is independent of σ^2.

Finally, since $\mathbb{E}[\mu^n] = \mathbb{E}[\theta] = 0$ and $\mathbb{E}[\mu^n\Theta] = \mathbb{E}\mathbb{E}\left[\mu^n\Theta \mid \Theta \in \left[\theta_i, \theta_{i+1}\right]\right] = \mathbb{E}\left[(\mu^n)^2\right] = var(\mu^n)$, we have

$$\begin{aligned}\mathbb{E}[u_R^{com}\left(y_R, \Theta, a, n\right)] &= -\mathbb{E}\left[\left(a\mu^n - a\Theta\right)^2\right] = -a^2\mathbb{E}\left[(\mu^n)^2 - 2\mu^n\Theta + \Theta^2\right] \\ &= a^2\left(var(\mu^n) - \sigma^2\right) = -a^2\left(1 - \ell(n, a, \psi)\right)\sigma^2.\end{aligned}$$

ii) $\mathbb{E}[u_R^{del}\left(y_S, \Theta, a\right)] = \mathbb{E}\left[-(\Theta - a\Theta)^2\right] = -(1-a)^2\sigma^2$. □

Proof of Proposition 2. The proof proceeds in three steps. In Step a), we compare a partition in the quantile space under distribution f to the same partition in the quantile space under distribution g. We start with $a = 1$ and then extend the comparison to $0 < a < 1$. In Step b), we consider a (partial) quantile partition which features a combination of f and g. In Step c), we combine Steps a) and b) and use an iterative procedure to derive an equilibrium partition out of the (partial) partition. This allows us to rank the equilibrium quantiles under f and g.

Step a) Let $h \in \{f_+, g_+\}$ and $H \in \{F_+, G_+\}$. As in Jewitt (1989) by a change of variables, the conditional expectation can be rewritten as

$$\mu_{i+1} = \mathbb{E}[\Theta \mid \Theta \in (t_i, t_{i+1})] = \int_{t_i}^{t_{i+1}} \theta\frac{h(\theta)}{H\left(t_{i+1}\right) - H\left(t_i\right)}d\theta = \int_{q_i}^{q_{i+1}} \frac{H^{-1}(z)}{q_{i+1} - q_i}dz$$

with $q_{i+1} = H\left(t_{i+1}\right)$ and $q_i = H\left(t_i\right)$.

Define $Q_H(q_i, q_{i+1}) := H(\mu_{i+1}) = H\left(\mathbb{E}\left[\Theta \mid H^{-1}(q_i) \leq \Theta \leq H^{-1}(q_{i+1})\right]\right)$.

Claim. *The convex transform order implies an order of the quantiles of the conditional expectations: If $G_+^{-1}F_+(\theta)$ is convex, then $Q_{F_+}(q_i, q_{i+1}) \leq Q_{G_+}(q_i, q_{i+1})$ for all q_i, q_{i+1}, $q_i \leq q_{i+1}$, $i = 1, \dots, n-1$.*

Proof. Assume $G_+^{-1}F_+(\theta)$ is convex. Jensen's inequality implies

$$G_+^{-1}F_+\left(\int_{q_i}^{q_{i+1}} F_+^{-1}(z)\frac{1}{F_+(F_+^{-1}(q_{i+1})) - F_+(F_+^{-1}(q_i))}dz\right) \leq \int_{q_i}^{q_{i+1}} G_+^{-1}F_+F_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz = \int_{q_i}^{q_{i+1}} G_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz.$$

Monotonicity of G_+ implies that

$$F_+\left(\int_{q_i}^{q_{i+1}} F_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz\right) \leq G_+\left(\int_{q_i}^{q_{i+1}} G_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz\right).$$

This is equivalent to

$$F_+\left(\mathbb{E}\left[\Theta | F_+^{-1}(q_i) \leq \Theta \leq F_+^{-1}(q_{i+1})\right]\right) \leq G_+\left(\mathbb{E}\left[\Theta | G_+^{-1}(q_i) \leq \Theta \leq G_+^{-1}(q_{i+1})\right]\right).$$

Thus, $Q_{F_+}(q_i, q_{i+1}) \leq Q_{G_+}(q_i, q_{i+1})$. □

Recall that the equilibrium thresholds satisfy $t_i^n - a \cdot \mu_i^n = a \cdot \mu_{i+1}^n - t_i^n$, for $i = 1, \dots, n$. This can be written as $t_i^n = \frac{a}{2} \cdot (\mu_i^n + \mu_{i+1}^n)$. For now, take $a = 1$.

Applying Jensen's inequality twice, we obtain

$$\begin{aligned}
&G_+^{-1}F_+\left(\frac{1}{2}\int_{q_{i-1}}^{q_i} \frac{F_+^{-1}(z)}{F_+(F_+^{-1}(q_i)) - F_+(F_+^{-1}(q_{i-1}))}dz + \frac{1}{2}\int_{q_i}^{q_{i+1}} \frac{F_+^{-1}(z)}{F_+(F_+^{-1}(q_{i+1})) - F_+(F_+^{-1}(q_i))}dz\right) \\
&\leq \frac{1}{2}G_+^{-1}F_+\left(\int_{q_{i-1}}^{q_i} F_+^{-1}(z)\frac{1}{q_i - q_{i-1}}dz\right) + \frac{1}{2}G_+^{-1}F_+\left(\int_{q_i}^{q_{i+1}} F_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz\right) \\
&\leq \frac{1}{2}\int_{q_{i-1}}^{q_i} G_+^{-1}F_+F_+^{-1}(z)\frac{1}{q_i - q_{i-1}}dz + \frac{1}{2}\int_{q_i}^{q_{i+1}} G_+^{-1}F_+F_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz \\
&= \frac{1}{2}\int_{q_{i-1}}^{q_i} G_+^{-1}(z)\frac{1}{q_i - q_{i-1}}dz + \frac{1}{2}\int_{q_i}^{q_{i+1}} G_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz.
\end{aligned}$$

Hence

$$\begin{aligned}
&F_+\left(\frac{1}{2}\int_{q_{i-1}}^{q_i} F_+^{-1}(z)\frac{1}{q_i - q_{i-1}}dz + \frac{1}{2}\int_{q_i}^{q_{i+1}} F_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz\right) \\
&\leq G_+\left(\frac{1}{2}\int_{q_{i-1}}^{q_i} G_+^{-1}(z)\frac{1}{q_i - q_{i-1}}dz + \frac{1}{2}\int_{q_i}^{q_{i+1}} G_+^{-1}(z)\frac{1}{q_{i+1}-q_i}dz\right).
\end{aligned}$$

Define the functions $v(q_i) := \frac{1}{2}\left(\frac{1}{q_i - q_{i-1}}\int_{q_{i-1}}^{q_i} F_+^{-1}(z)dz + \frac{1}{q_{i+1}-q_i}\int_{q_i}^{q_{i+1}} F_+^{-1}(z)dz\right)$ and $z(q_i) := \frac{1}{2}\left(\frac{1}{q_i - q_{i-1}}\int_{q_{i-1}}^{q_i} G_+^{-1}(z)dz + \frac{1}{q_{i+1}-q_i}\int_{q_i}^{q_{i+1}} G_+^{-1}(z)dz\right)$.

Then the inequality can be written as

$$G_+\left(\frac{z(q_i)}{v(q_i)}v(q_i)\right) \geq F_+(v(q_i)) \text{ for all } q_i \in [q_{i-1}, q_{i+1}].$$

Applying the inverse of G^{-1} and dividing by $v(q_i)$, this is equivalent to

$$\frac{z(q_i)}{v(q_i)} \geq \frac{G_+^{-1}F_+(v(q_i))}{v(q_i)} \text{ for all } q_i \in [q_{i-1}, q_{i+1}]. \tag{8}$$

We next want to introduce $a \in (0,1)$. We aim at showing that

$$G_+\left(a\frac{z\left(q_i\right)}{v\left(q_i\right)}v\left(q_i\right)\right) \geq F_+\left(av\left(q_i\right)\right) \text{ for all } a \in (0,1) \text{ and all } q_i \in \left[q_{i-1}, q_{i+1}\right].$$

Applying the inverse of G^{-1} and dividing by $av\left(q_i\right)$, this is equivalent to

$$\frac{z\left(q_i\right)}{v\left(q_i\right)} \geq \frac{G_+^{-1}F_+\left(av\left(q_i\right)\right)}{av\left(q_i\right)}.$$

This is equivalent to (8) for $a = 1$. Moreover, note that the convex transform order implies the star order (see Shaked and Shanthikumar (2007), p. 214): $G_+^{-1}F_+(\theta)$ convex implies that $\frac{G_+^{-1}F_+(\theta)}{\theta}$ increases in θ.

To apply this order to our condition, note that $av(u)$ increases in a and ranges from 0 to $v\left(q_i\right)$ for $a \in [0,1]$. Hence, setting $a < 1$ reduces the value of the right side of the inequality, and since the inequality holds for $a = 1$, it continues to hold for $a < 1$.

Step b) Recall that in the quantile space, the equilibrium condition $t_i^n = \frac{a}{2}\cdot(\mu_i^n + \mu_{i+1}^n)$ can be written as

$$q_{i,h} = H_+\left(\frac{a}{2}\left(\frac{1}{q_{i,h}-q_{i-1,h}}\int_{q_{i-1,h}}^{q_{i,h}} H_+^{-1}(z)\,dz + \frac{1}{q_{i+1,h}-q_{i,h}}\int_{q_{i,h}}^{q_{i+1,h}} H_+^{-1}(z)\,dz\right)\right),$$

for $h = f, g$ and $H = F, G$.

Fix the equilibrium thresholds $q_{i-1,f}$ and $q_{i+1,f}$, and consider $q_i = q_{i,gf}$ as the following function that combines F and G

$$G_+\left(\frac{a}{2}\left(\frac{1}{q_i-q_{i-1,f}}\int_{q_{i-1,f}}^{q_i} G_+^{-1}(z)\,dz + \frac{1}{q_{i+1,f}-q_i}\int_{q_i}^{q_{i+1,f}} G_+^{-1}(z)\,dz\right)\right).$$

By Jensen's inequality, we have

$$G_+\left(\frac{1}{2}\left(\frac{1}{q_i-q_{i-1,f}}\int_{q_{i-1,f}}^{q_i} G_+^{-1}(z)\,dz + \frac{1}{q_{i+1,f}-q_i}\int_{q_i}^{q_{i+1,f}} G_+^{-1}(z)\,dz\right)\right)$$

$$\geq F_+\left(\frac{1}{2}\left(\frac{1}{q_i-q_{i-1,f}}\int_{q_{i-1,f}}^{q_i} F_+^{-1}(z)\,dz + \frac{1}{q_{i+1,f}-q_i}\int_{q_i}^{q_{i+1,f}} F_+^{-1}(z)\,dz\right)\right),$$

for all $q_i \in \left[q_{i-1,f}, q_{i+1,f}\right]$. Thus the same inequality holds in particular at $q_{i,f}$. The fact that $\frac{G_+^{-1}F(\theta)}{\theta}$ is increasing in θ implies that

$$G_+\left(\frac{a}{2}\left(\frac{1}{q_i-q_{i-1,f}}\int_{q_{i-1,f}}^{q_i} G_+^{-1}(z)\,dz + \frac{1}{q_{i+1,f}-q_i}\int_{q_i}^{q_{i+1,f}} G_+^{-1}(z)\,dz\right)\right)$$

$$\geq F_+\left(\frac{a}{2}\left(\frac{1}{q_i-q_{i-1,f}}\int_{q_{i-1,f}}^{q_i} F_+^{-1}(z)\,dz + \frac{1}{q_{i+1,f}-q_i}\int_{q_i}^{q_{i+1,f}} F_+^{-1}(z)\,dz\right)\right), \tag{9}$$

for all $q_i \in \left[q_{i-1,f}, q_{i+1,f}\right]$.

Since condition (9) holds for any arbitrary (quantile) threshold q_i, it holds for all $i = 1, \ldots, n$.

Step c) Denote the equilibrium quantile partition under f_+, $q_{i,f}^n$, $i = 1, \ldots, n$, as

$$q_{i,f}^n = F_+\left(\frac{a}{2}\left(\frac{1}{q_{i,f}^n-q_{i-1,f}^n}\int_{q_{i-1,f}^n}^{q_{i,f}^n} F_+^{-1}(z)\,dz + \frac{1}{q_{i+1,f}^n-q_{i,f}^n}\int_{q_{i,f}^n}^{q_{i+1,f}^n} F_+^{-1}(z)\,dz\right)\right)$$

for all $i = 1, \ldots, n$. By convention, $q_{0,f}^n = 0$ and $q_{n+1,f}^n = 1$.

By Steps a) and b), we therefore have

$$q_{i,f}^n \leq G_+\left(\frac{a}{2}\left(\frac{1}{q_{i,f}^n - q_{i-1,f}^n}\int_{q_{i-1,f}^n}^{q_{i,f}^n} G_+^{-1}(z)\,dz + \frac{1}{q_{i+1,f}^n - q_{i,f}^n}\int_{q_{i,f}^n}^{q_{i+1,f}^n} G_+^{-1}(z)\,dz\right)\right).$$

Let $t_{i,gf} := G_+^{-1}\left(q_{i,f}^n\right)$. Then

$$t_{i,gf} \leq \frac{a}{2}\left(\mu_{i-1,g}\left(t_{i-1,gf}, t_{i,gf}\right) + \mu_{i,g}\left(t_{i,gf}, t_{i+1,gf}\right)\right). \tag{10}$$

It follows from this inequality, that for any fixed $t_{i-1,gf}$ and $t_{i+1,gf}$, the value of $t_i = t_{i,gf}$ is too low to be part of an equilibrium.

Given this observation, we consider the following iterative procedure: For any fixed $t_{i-1,gf}$, we denote the "partial equilibrium thresholds" under g by $t_{j,g}^{(*)}$ for all $j \geq i$, where the distribution is adjusted from f to g on the entire support, the equilibrium thresholds above $t_{i-1,gf}$ are adjusted to g, $t_j = t_{j,g}^{(*)}$ for $j \geq i$, but the equilibrium thresholds below $t_{i-1,gf}$ and not adjusted, $t_j = t_{j,gf}$ for $j < i$.

At iteration step one, keep all thresholds $t_i = t_{i,gf}$ for $i = 1, \dots, n-1$ fixed and let t_n adjust to $t_{n,g}^{(*)} = t_{n,g}^{(*)}\left(t_{n-1,gf}\right)$. At $t_{n,g}^{(*)}$, the sender is indifferent under g between pooling upwards or downwards given that the receiver best replies with respect to g.

At iteration step l, keep all thresholds $t_i = t_{i,gf}$ for $i = 1, \dots, n-l$ fixed, adjust threshold t_{n-l+1} to make the sender indifferent at $t_{n-l+1,g}^{(*)} = t_{n-l+1,g}^{(*)}\left(t_{n-l,gf}\right)$, and keep the sender indifferent at all thresholds $t_{j,g}^{(*)}$ for $j \geq n-l+2$. Note that all $t_{j,g}^{(*)}$ depend recursively on the initial value $t_{n-l,gf}$ and on their respective predecessors $t_{n-l+1,g}^{(*)}, \dots, t_{j-1,g}^{(*)}$.

At iteration step one, we observe that by (10), for $t_n = t_{n,gf}$,

$$t_n - a\mu_{n,g}\left(t_{n-1,gf}, t_n\right) \leq a\mu_{n+1,g}\left(t_n, \overline{S}_g\right) - t_n.$$

By logconcavity of the density, $\mu_{n+1,g}\left(t_n, \overline{S}_g\right)$ and $\mu_{n,g}\left(t_{n-1,gf}, t_n\right)$ each increase in t_n less than one for one. Hence, there exists a unique $t_{n,g}^{(*)} \geq t_{n,gf}$ such that

$$t_{n,g}^{(*)} - a\mu_{n,g}(t_{n-1,gf}, t_{n,g}^{(*)}) = a\mu_{n+1,g}(t_{n,g}^{(*)}, \overline{S}_g) - t_{n,g}^{(*)}. \tag{11}$$

Consider an arbitrary iteration step $l < n$. Suppose that all thresholds $t_{j,g}^{(*)}$ for $j = l+1, \dots, n$ have been adjusted "weakly upwards".

Since increasing thresholds increases the right side of (10), the inequality continues to hold. It remains to be shown that there is a unique $t_l = t_{l,g}^{(*)}$ such that

$$\left(a\mu_{l+1,g}(t_l, t_{l+1,g}^{(*)}) - t_l\right) - \left(t_l - a\mu_{l,g}\left(t_{l-1,gf}, t_l\right)\right) = 0. \tag{12}$$

Differentiating the left side of (12) with respect to t_l, we get

$$-2 + a\frac{\partial}{\partial t_l}\mu_{l,g}\left(t_{l-1,gf}, t_l\right) + a\frac{\partial}{\partial t_l}\mu_{l+1,g}(t_l, t_{l+1,g}^{(*)}) + a\frac{\partial}{\partial t_{l+1}}\mu_{l+1,g}(t_l, t_{l+1,g}^{(*)})\frac{dt_{l+1,g}^{(*)}}{dt_l}.$$

By logconcavity, $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$ implies that this expression is negative. We show that $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$ holds by induction: Totally differentiating (11) with respect to $t_{n,g}^{(*)}$ and $t_{n-1,gf}$, we find that

$$\frac{dt_{n,g}^{(*)}}{dt_{n-1,gf}} = \frac{a\frac{\partial}{\partial t_{n-1,gf}}\mu_{n,g}\left(t_{n-1,gf}, t_{n,g}^{(*)}\right)}{2 - a\frac{\partial}{\partial t_{n,g}^{(*)}}\mu_{n,g}\left(t_{n-1,gf}, t_{n,g}^{(*)}\right) - a\frac{\partial}{\partial t_{n,g}^{(*)}}\mu_{n+1,g}\left(t_{n,g}^{(*)}, \overline{S}_g\right)} \leq 1,$$

where the inequality is due to logconcavity of the density.

Next, suppose that $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$. Totally differentiating (12) we get

$$\frac{dt_{l,g}^{(*)}}{dt_{l-1,gf}} = \frac{a\frac{\partial}{\partial t_{l-1,gf}}\mu_{l,g}(t_{l-1,gf}, t_{l,g}^{(*)})}{2 - a\frac{\partial}{\partial t_{l,g}^{(*)}}\mu_{l,g}(t_{l-1,gf}, t_{l,g}^{(*)}) - a\frac{\partial}{\partial t_{l,g}^{(*)}}\mu_{l+1,g}(t_{l,g}^{(*)}, t_{l+1,g}^{(*)}) - a\frac{\partial}{\partial t_{l+1,g}^{(*)}}\mu_{l+1,g}(t_{l,g}^{(*)}, t_{l+1,g}^{(*)})\frac{dt_{l+1,g}^{(*)}}{dt_{l,g}^{(*)}}} \leq 1,$$

by logconcavity of the density and the assumption that $\frac{dt_{l+1}^{(*)}}{dt_l} \leq 1$. This concludes the argument.

Switching back to quantiles, we have demonstrated $t^n_{i,g} \geq t_{i,gf} = G^{-1}_+\left(q^n_{i,f}\right)$, and hence

$$G_+\left(t^n_{i,g}\right) \geq q^n_{i,f} = F_+\left(t^n_{i,f}\right) \text{ for all } i. \quad \square$$

Proof of Proposition 3. Assume that the equilibrium partition under distribution g, $t^n_{i,g}$, satisfies the following condition,

$$\begin{aligned}&\mathbb{E}_f\left[\Theta_f\middle|\Theta_f \in \left[t^n_{i-1,g}, t^n_{i,g}\right]\right] + \mathbb{E}_f\left[\Theta_f\middle|\Theta_f \in \left[t^n_{i,g}, t^n_{i+1,g}\right]\right]\\ &> \mathbb{E}_g\left[\Theta_g\middle|\Theta_g \in \left[t^n_{i-1,g}, t^n_{i,g}\right]\right] + \mathbb{E}_g\left[\Theta_g\middle|\Theta_g \in \left[t^n_{i,g}, t^n_{i+1,g}\right]\right],\end{aligned} \tag{13}$$

where $t^n_{n+1,g} = \overline{S}_g$ and $t^n_{0,g} = 0$.

To prove the proposition, we need to show that quantiles and receiver induced actions are more risky in the sense of a mean-variance-preserving spread under distribution F than under distribution G. Recall from the proof of Proposition 2 that $\Theta_{f+} \leq_c \Theta_{g+}$ implies that the quantiles satisfy $F_+\left(t^n_{i,f}\right) \leq G_+\left(t^n_{i,g}\right)$ for all i. Thus, to prove the proposition, it suffices to order the receiver's induced actions as well. In particular, we show that condition (13) implies that under distribution f the equilibrium critical types are strictly higher and strictly better for the sender than the equilibrium critical types under distribution g. Finally, we will show that the conditions stated in Proposition 3 are sufficient for (13) to hold.

As Fig. 6 and the uniform conditional variability order, $\Theta_{f_+} \leq_{uv} \Theta_{g_+}$, reveal, the local stochastic order depends on the location of the equilibrium thresholds considered. By symmetry, we focus on the positive half of the support only. For intervals below (above) the mode m, the truncated distributions under f_+ dominate (are dominated by) the truncated distributions under g_+ in the likelihood ratio order. To have some control over which order applies to which partition intervals – for example, to the first n intervals – it is helpful to rely on monotonicity of equilibria in the alignment parameter a:

Claim A.1. *For any symmetric logconcave density and for any n, the equilibrium critical types $t^n_i(a)$ and induced means $\mu^n_i(a)$ are strictly increasing in a for all i.*

For a proof see, e.g., Deimen and Szalay (2023) or Chen and Gordon (2015). Moreover, we note that $a \to 0$ implies that $t^n_i(a) \to 0$ for $i = 1, \ldots, n$.

The proof of Proposition 3, is completed through the following sequence of lemmas that show that condition (13) is satisfied.

Lemma A.4. *(Metzger and Rüschendorf (1991))*
Let $\frac{f_+(\theta)}{g_+(\theta)}$ be unimodal with interior mode m. The function $\frac{F_+(x)}{G_+(x)}$ inherits unimodality with mode $m_1 > m$, the function $\frac{(1-F_+(x))}{(1-G_+(x))}$ inherits unimodality with mode $m_2 < m$. Moreover, there exists a unique $\hat{x}$ such that $F_+(\theta) < G_+(\theta)$ for $\theta \in (0, \hat{x})$, $F_+(\hat{x}) = G_+(\hat{x})$, and $F_+(\theta) > G_+(\theta)$ for $\theta \in (\hat{x}, \infty)$.

Proof. Metzger and Rüschendorf (1991) Section 2. $\square$

For the following lemma, since $\int_x^{\overline{S}_h}\left(1 - H_+(\theta)\right) d\theta = \int_x^{\infty}\left(1 - H_+(\theta)\right) d\theta$ as $H_+(\theta) = 1$ for $\theta \geq \overline{S}_h$, we unify notation and write $\int_x^\infty$ for infinite as well as for finite supports, $[0, \overline{S}_h]$.

Lemma A.5. *(i) Let m denote the mode of the function $\frac{f_+(\theta)}{g_+(\theta)}$. Conditional on $\theta \in [0, m)$, the distributions f_+ and g_+ satisfy the monotone likelihood ratio property.*

(ii) The function $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ is unimodal in $x \in [0, \overline{S}_f]$ with mode $m' \in (0, m_2)$;

for $0 \leq x \leq (<) m'$, we have $\mathbb{E}\left[\Theta_f\middle|\Theta_f \geq x\right] \geq (>) \mathbb{E}\left[\Theta_g\middle|\Theta_g \geq x\right]$.

Proof of Lemma A.5. (i) Follows from the proof of Lemma 4.

(ii) We first show that $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ is unimodal with mode m'. We then show that the mode m' is interior.

Straightforward differentiation gives

$$\frac{\partial}{\partial x}\frac{\int_x^\infty \left(1-F_+(\theta)\right)d\theta}{\int_x^\infty \left(1-G_+(\theta)\right)d\theta}=\frac{-\left(1-F_+(x)\right)\int_x^\infty \left(1-G_+(\theta)\right)d\theta+\left(1-G_+(x)\right)\int_x^\infty \left(1-F_+(\theta)\right)d\theta}{\left(\int_x^\infty \left(1-G_+(\theta)\right)d\theta\right)^2}.$$

The sign of the derivative is positive if and only if

$$\left(1-F_+(x)\right)\int_x^\infty \left(1-G_+(\theta)\right)d\theta<\left(1-G_+(x)\right)\int_x^\infty \left(1-F_+(\theta)\right)d\theta.$$

Note that by an integration by parts for any $x\in[0,\overline{S}_h)$, we have that for $h_+\in\left\{f_+,g_+\right\}$ and $H_+\in\left\{F_+,G_+\right\}$

$$\mathbb{E}\left[\Theta|\,\Theta\geq x\right]=\frac{\int_x^\infty \theta h_+(\theta)\,d\theta}{1-H_+(x)}=x+\frac{\int_x^\infty \left(1-H_+(\theta)\right)d\theta}{1-H_+(x)}.$$

Hence, $\frac{\partial}{\partial x}\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}\gtreqless 0$ if and only if $\mathbb{E}\left[\Theta_f\middle|\Theta_f\geq x\right]\gtreqless\mathbb{E}\left[\Theta_g\middle|\Theta_g\geq x\right]$.

Since a mode is an extremum, it is either at the boundary or satisfies the first order condition $\mathbb{E}\left[\Theta_f\middle|\Theta_f\geq x^*\right]=\mathbb{E}\left[\Theta_g\middle|\Theta_g\geq x^*\right]$. We next prove that there is at most one such value $x^*=m'$.

By Lemma A.4, the function $\frac{(1-F_+(x))}{(1-G_+(x))}$ is unimodal with mode m_2. Thus for $x\geq m_2$ the function is decreasing, equivalent to the conditional distribution of Θ_g conditional on $\Theta_g\geq x$ under distribution G_+ first order stochastically dominating the conditional distribution of Θ_f conditional on $\Theta_f\geq x$ under F_+: for $x\geq m_2$,

$$\frac{1-F_+(x)}{1-G_+(x)}>\frac{1-F_+(\theta)}{1-G_+(\theta)}\quad\Leftrightarrow\quad\frac{F_+(\theta)-F_+(x)}{1-F_+(x)}>\frac{G_+(\theta)-G_+(x)}{1-G_+(x)}.$$

By implication, for $x\geq m_2$ we have $\mathbb{E}\left[\Theta_f\middle|\Theta_f\geq x\right]<\mathbb{E}\left[\Theta_g\middle|\Theta_g\geq x\right]$ and $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ is strictly decreasing.

For $x^*<m_2$, recall that by the first order condition we have

$$-\left(1-F_+\left(x^*\right)\right)\int_{x^*}^\infty \left(1-G_+(\theta)\right)d\theta+\left(1-G_+\left(x^*\right)\right)\int_{x^*}^\infty \left(1-F_+(\theta)\right)d\theta=0.$$

Differentiating a second time and evaluating at x^*, we get

$$f_+\left(x^*\right)\int_{x^*}^\infty \left(1-G_+(\theta)\right)d\theta-g_+\left(x^*\right)\int_{x^*}^\infty \left(1-F_+(\theta)\right)d\theta$$

$$<g_+\left(x^*\right)\frac{1-F_+(x)}{\left(1-G_+(x)\right)}\int_{x^*}^\infty \left(1-G_+(\theta)\right)d\theta-g_+\left(x^*\right)\int_{x^*}^\infty \left(1-F_+(\theta)\right)d\theta=0,$$

where the equality follows from the first order condition. For the inequality note that the function $\frac{(1-F_+(x))}{(1-G_+(x))}$ is increasing if and only if the hazard rates of the distributions satisfy

$$\frac{f_+(x)}{1-F_+(x)}<\frac{g_+(x)}{\left(1-G_+(x)\right)},$$

thus for $x<m_2$. The second derivative being negative implies that any stationary point must be a maximum, hence there is at most one such point m'.

Finally, we prove that the mode m' of $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$ must be interior. For contradiction suppose that m' is at the boundary. From the first part of the proof, $m' \le m_2$, so that m' cannot be at the upper end of the support. Thus suppose that $m' = 0$, so that $\frac{\partial}{\partial x}\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta} < 0$ for all $x \in [0, \overline{S}_f]$.

The variance of the distribution over the whole support (positive and negative) can by symmetry ($h_+ = 2h$) and by integrating by parts twice be written as

$$\int_{-\infty}^{\infty} \theta^2 h(\theta)\, d\theta = \int_0^{\infty} \theta^2 h_+(\theta)\, d\theta = 2\int_0^{\infty} \theta\left(1 - H_+(\theta)\right) d\theta = 2\int_0^{\infty}\int_x^{\infty}\left(1 - H_+(\theta)\right) d\theta dx,$$

with $h \in \{f, g\}$, $h_+ \in \{f_+, g_+\}$, and $H_+ \in \{F_+, G_+\}$.

We can further rewrite and integrate by parts to obtain

$$2\int_0^{\infty}\int_x^{\infty}\left(1 - F_+(\theta)\right) d\theta dx = 2\int_0^{\infty} \frac{\int_x^{\infty}\left(1 - F_+(\theta)\right) d\theta}{\int_x^{\infty}\left(1 - G_+(\theta)\right) d\theta}\int_x^{\infty}\left(1 - G_+(\theta)\right) d\theta dx$$

$$= -2\frac{\int_z^{\infty}\left(1 - F_+(\theta)\right) d\theta}{\int_z^{\infty}\left(1 - G_+(\theta)\right) d\theta}\int_z^{\infty}\int_x^{\infty}\left(1 - G_+(\theta)\right) d\theta dx\Bigg|_0^{\infty} + 2\int_0^{\infty}\frac{\partial}{\partial z}\frac{\int_z^{\infty}\left(1 - F_+(\theta)\right) d\theta}{\int_z^{\infty}\left(1 - G_+(\theta)\right) d\theta}\int_z^{\infty}\int_x^{\infty}\left(1 - G_+(\theta)\right) d\theta dx dz$$

$$= 2\frac{\int_0^{\infty}\left(1 - F_+(\theta)\right) d\theta}{\int_0^{\infty}\left(1 - G_+(\theta)\right) d\theta}\int_0^{\infty}\int_x^{\infty}\left(1 - G_+(\theta)\right) d\theta dx + 2\int_0^{\infty}\frac{\partial}{\partial z}\frac{\int_z^{\infty}\left(1 - F_+(\theta)\right) d\theta}{\int_z^{\infty}\left(1 - G_+(\theta)\right) d\theta}\int_z^{\infty}\int_x^{\infty}\left(1 - G_+(\theta)\right) d\theta dx dz.$$

Substituting for $\mu_{h_+} = \int_0^{\infty}(1 - H(\theta))\, d\theta$ and $\sigma_h^2 = 2\int_0^{\infty}\int_x^{\infty}\left(1 - H_+(\theta)\right) d\theta dx$, we have that

$$\sigma_f^2 - \frac{\mu_{f_+}}{\mu_{g_+}}\sigma_g^2 = 2\int_0^{\infty}\frac{\partial}{\partial z}\frac{\int_z^{\infty}\left(1 - F_+(\theta)\right) d\theta}{\int_z^{\infty}\left(1 - G_+(\theta)\right) d\theta}\int_z^{\infty}\int_x^{\infty}\left(1 - G_+(\theta)\right) d\theta dx dz.$$

We have that $m' = 0$ implies $\frac{\mu_{f+}}{\mu_{g+}} \le 1$. Moreover, by assumption $\sigma_f^2 = \sigma_g^2$. Hence the left side is non-negative. However, the right side is strictly negative due to our contradictory hypothesis that $\frac{\partial}{\partial z}\frac{\int_z^\infty (1-F_+(\theta))d\theta}{\int_z^\infty (1-G_+(\theta))d\theta} < 0$ for all $z \in [0, \overline{S}_f]$. □

To complete the proof, we note that Lemma A.5 implies that (13) applies for a sufficiently low. This in turn implies that for a fixed sender partition $(t_{i,g}^n)$, the receiver's induced actions are higher under f_+ than under g_+. Hence, the equilibrium under f_+ needs to feature higher receiver equilibrium induced actions:

Lemma A.6. *For any two symmetric, logconcave densities f, g with the same variance and with truncated densities f_+, g_+ that satisfy $\Theta_{f_+} \le_{uv} \Theta_{g_+}$, there exists a unique a' such that $\mathbb{E}\left[\Theta_f \middle| \Theta_f \ge t_{n,g}^n(a')\right] = \mathbb{E}\left[\Theta_g | \Theta_g \ge t_{n,g}^n(a')\right]$. Moreover, for $a < a'$, all $n+1$ receiver equilibrium actions under distribution f_+ are strictly higher than under g_+, $a \cdot \mu_f\left(t_{i-1,f}^n, t_{i,f}^n\right) > a \cdot \mu_g^n\left(t_{i-1,g}^n, t_{i,g}^n\right)$ for all i.*

Proof of Lemma A.6. By Lemma A.5, the tail-truncated expectation functions, $\mathbb{E}\left[\Theta_f \middle| \Theta_f \ge x\right]$ and $\mathbb{E}\left[\Theta_g \middle| \Theta_g \ge x\right]$, cross exactly once in the interior of the positive half of the support. The intersection is at $x = m'$, the mode of the ratio $\frac{\int_x^\infty (1-F_+(\theta))d\theta}{\int_x^\infty (1-G_+(\theta))d\theta}$. Hence,

$\mathbb{E}\left[\Theta_f \middle| \Theta_f \geq t^n_{n,g}(a)\right] \geq \mathbb{E}\left[\Theta_g \middle| \Theta_g \geq t^n_{n,g}(a)\right]$ if and only if $t^n_{n,g}(a) \leq m'$. By Claim A.1, $t^n_{n,g}(a)$ is strictly increasing in a, so by continuity there is a unique a' such that $t^n_{n,g}(a') = m'$ and moreover, $t^n_{n,g}(a) < m'$ for $a < a'$.

By Lemma A.5, the distributions below $t^n_{n,g}(a)$ satisfy that $f_+(\theta)/g_+(\theta)$ increasing in θ for all $\theta \leq m$ if $t^n_{n,g}(a) \leq m$. By Lemma A.5, $m' < m_2$. By Lemma A.4, $m_2 < m$. Hence, $a \leq a'$ implies that $f_+(\theta)/g_+(\theta)$ is increasing for all $\theta \leq t^n_{n,g}(a)$. Since the monotone likelihood ratio property is preserved under multiplication of a constant, the truncated distribution below $t^n_{n,g}(a)$ satisfies the monotone likelihood ratio property, $\frac{\partial}{\partial\theta}\frac{f_+(\theta)}{F_+(t^n_{n,g}(a))} / \frac{g_+(\theta)}{G_+(t^n_{n,g}(a))} > 0$. More generally, the conditional distributions truncated to any interval $\left[t^n_{i-1,g}(a), t^n_{i,g}(a)\right)$ satisfy $\frac{\partial}{\partial\theta}\frac{f_+(\theta)}{F_+\left(t^n_{i,g}(a)\right)-F_+\left(t^n_{i-1,g}(a)\right)} / \frac{g_+(\theta)}{G_+\left(t^n_{i,g}(a)\right)-G_+\left(t^n_{i-1,g}(a)\right)} > 0$ for $i = 1, \ldots, n$. As is well known, the monotone likelihood ratio property implies first order stochastic dominance, which in turn implies that inequality (13) is satisfied for all $i = 1, \ldots, n$ if we keep the partition at the equilibrium partition under g_+, $(t^n_{i,g})$. Therefore, we can conclude that both the equilibrium critical types and the receiver's induced actions are increased so that $\mu_f\left(t^n_{i-1,f}(a), t^n_{i,f}(a)\right) \geq \mu^n_g\left(t^n_{i-1,g}(a), t^n_{i,g}(a)\right)$ for $i = 1, \ldots, n$ for $a \leq a'$. □

□

Proof of Proposition 4. The proof of the second part regarding linear tail-truncated expectations is given in Deimen and Szalay (2019). The proof of the first part extends that proof to convex tail-truncated expectations. Before proving the result by induction, we make some preliminary observations on the conditional probabilities and the tail-truncated expectation function. A more detailed version of the proof can be found in the working paper version Deimen and Szalay (2023).

For $k = 2, \ldots, n$, define $\hat{p}_{k-1}$ as the probability that $\theta \in \left[t_{k-2}, t_{k-1}\right]$ conditional on $\theta \geq t_{k-2}$,

$$\hat{p}_{k-1} \equiv \frac{F_+\left(t_{k-1}\right) - F_+\left(t_{k-2}\right)}{1 - F_+\left(t_{k-2}\right)}.$$

Accordingly, $1 - \hat{p}_{k-1} = \frac{1-F_+\left(t_{k-1}\right)}{1-F_+\left(t_{k-2}\right)}$ is the probability that $\theta \geq t_{k-1}$, conditional on $\theta \geq t_{k-2}$. Note that $\hat{p}_{k-1}\mu_{k-1} = \mathbb{E}\left[\theta \middle| \theta \geq t_{k-2}\right] - \left(1 - \hat{p}_{k-1}\right)\mathbb{E}\left[\theta \middle| \theta \geq t_{k-1}\right]$. Solving for $\hat{p}_{k-1}$, we can write the probabilities as

$$\hat{p}_{k-1} = \frac{\mathbb{E}\left[\theta \middle| \theta \geq t_{k-1}\right] - \mathbb{E}\left[\theta \middle| \theta \geq t_{k-2}\right]}{\mathbb{E}\left[\theta \middle| \theta \geq t_{k-1}\right] - \mu_{k-1}} \quad \text{and} \quad 1 - \hat{p}_{k-1} = \frac{\mathbb{E}\left[\theta \middle| \theta \geq t_{k-2}\right] - \mu_{k-1}}{\mathbb{E}\left[\theta \middle| \theta \geq t_{k-1}\right] - \mu_{k-1}}.$$

Observe that $\left(1 - \hat{p}_{k-2}\right) \cdot \hat{p}_{k-1}$ is the probability of the event $\theta \in \left[t_{k-2}, t_{k-1}\right]$ conditional on $\theta \geq t_{k-3}$, and $\left(1 - \hat{p}_{k-2}\right) \cdot \left(1 - \hat{p}_{k-1}\right)$ is the probability of the event $\theta \geq t_{k-1}$ conditional on $\theta \geq t_{k-3}$. To see this, note that $1 - \hat{p}_{k-2} = \Pr\left[\theta \geq t_{k-2} \middle| \theta \geq t_{k-3}\right] = \frac{1-F_+\left(t_{k-2}\right)}{1-F_+\left(t_{k-3}\right)}$ and recall that $\hat{p}_{k-1} = \frac{F_+\left(t_{k-1}\right)-F_+\left(t_{k-2}\right)}{1-F_+\left(t_{k-2}\right)}$.

Define, for all $t > 0$

$$\alpha(t) := \frac{\mathbb{E}[\theta | \theta \geq t] - \mathbb{E}[\theta | \theta \geq 0]}{t} = \frac{\phi(t) - \phi(0)}{t}.$$

Define $\mu_+ := \phi(0) = \mathbb{E}[\theta | \theta \geq 0]$. Note that $\phi(t) = \mathbb{E}[\theta | \theta \geq t]$ can always be written as the *pseudo linear interpolation* $\mathbb{E}[\theta | \theta \geq t] = \mu_+ + t \cdot \alpha(t)$. In the case of a linear tail-truncated expectation, $\alpha(t)$ is a constant. In the convex case, we show that $\alpha(t)$ is increasing in t:

For $t = 0$, we take the limit $\alpha(0) = \lim_{t\to 0}\frac{\mathbb{E}[\theta|\theta\geq t]-\mu_+}{t} = \frac{\partial}{\partial t}\mathbb{E}[\theta | \theta \geq t]|_{t=0}$. Likewise, by l'Hôpital's rule, $\lim_{t\to\infty}\frac{\mathbb{E}[\theta|\theta\geq t]-\mu_+}{t} = \lim_{t\to\infty}\frac{\partial}{\partial t}\mathbb{E}[\theta | \theta \geq t]$. Moreover, $\alpha'(t) = \frac{\frac{\partial}{\partial t}\mathbb{E}[\theta|\theta\geq t]t - (\mathbb{E}[\theta|\theta\geq t]-\mu_+)}{t^2} = \frac{1}{t}\left(\frac{\partial}{\partial t}\mathbb{E}[\theta | \theta \geq t] - \alpha(t)\right)$. By the fundamental theorem of calculus $\alpha(t) = \frac{\mathbb{E}[\theta|\theta\geq t]-\mu_+}{t} = \frac{\int_0^t \frac{\partial}{\partial z}\mathbb{E}[\theta|\theta\geq z]dz}{t}$. By the intermediate value theorem for integrals, there is some value $t^* \in (0, t)$ such that $\frac{\int_0^t \frac{\partial}{\partial z}\mathbb{E}[\theta|\theta\geq z]dz}{t} = \frac{\partial}{\partial z}\mathbb{E}[\theta | \theta \geq z]\Big|_{z=t^*}$. Hence, $\alpha'(t) = \frac{1}{t}\left(\frac{\partial}{\partial z}\mathbb{E}[\theta | \theta \geq z]\Big|_{z=t} - \frac{\partial}{\partial z}\mathbb{E}[\theta | \theta \geq z]\Big|_{z=t^*}\right) \geq 0$, where the inequality follows from $t^* \in (0, t)$ and from convexity of $\mathbb{E}[\theta | \theta \geq t]$ in t. Thus, $\alpha(t)$ is increasing in t and hence minimal at $\alpha(0) =: \underline{\alpha}$.

Recall the alignment parameter $a \in (0, 1)$. Define $\hat{c} := \underline{\alpha} \cdot a$.

Assume that $\hat{c} \in (0, 2)$. Note that for all distributions with logconcave densities this is not a constraint. In this case, $\underline{\alpha} \leq \alpha(t) \leq 1$ for all t, since logconcave densities have a decreasing mean residual life (see Bagnoli and Bergstrom (2005), Theorem 3 and Lemma 2) and $\alpha(t) > 1$ for some $t > 0$ would imply that the mean residual life at t is higher than at zero, a contradiction.

Let $X^n_k\left(t^n_{k-1}\right)$ be equal to $\hat{c}^2$ times the expected squared deviation of the truncated means from μ_+, conditional on $\theta \geq t^n_{k-1}$,

$$X^n_k\left(t^n_{k-1}\right) := \hat{p}^n_k\left(\hat{c}\mu^n_k - \hat{c}\mu_+\right)^2 + \left(1 - \hat{p}^n_k\right)X^n_{k+1}(t_k).$$

Induction hypothesis.

$$X^n_k\left(t^n_{k-1}\right) \geq \underline{X}^n_k\left(t^n_{k-1}\right) := \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ + \hat{c}\mu^n_k\right)\left(\hat{c}\mu_+ - \hat{c}\mu^n_k\right) + 2\left(\hat{c}\mathbb{E}\left[\theta \middle| \theta \geq t^n_{k-1}\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu^n_k\right) - \hat{c}\mu_+\right).$$

Induction base. The proof of the induction base is a simplified version of the proof of the induction step and therefore omitted.
Induction step.

By definition, $X^n_{k-1}\left(t^n_{k-2}\right)=\hat{p}_{k-1}\left(\hat{c}\mu^n_{k-1}-\hat{c}\mu_+\right)^2+\left(1-\hat{p}_{k-1}\right)X^n_k\left(t^n_{k-1}\right)$. Since $X^n_k\left(t^n_{k-1}\right)\geq \underline{X}^n_k\left(t^n_{k-1}\right)$, we have

$$X^n_{k-1}\left(t^n_{k-2}\right)\geq \hat{p}_{k-1}\left(\hat{c}\mu^n_{k-1}-\hat{c}\mu_+\right)^2+\left(1-\hat{p}_{k-1}\right)\underline{X}^n_k\left(t^n_{k-1}\right)=:\widehat{X}^n_{k-1}\left(t^n_{k-2}\right).$$

Substituting for the probability $\hat{p}_{k-1}$ and for $\underline{X}^n_k\left(t^n_{k-1}\right)$, we obtain

$$\begin{aligned}\widehat{X}^n_{k-1}\left(t^n_{k-2}\right)=&\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\left(\hat{c}\mu^n_{k-1}-\hat{c}\mu_+\right)^2+\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t_{k-2}\right]-\hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\\&\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_++\hat{c}\mu^n_k\right)\left(\hat{c}\mu_+-\hat{c}\mu^n_k\right)+2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}\right]-\hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_++\mu^n_k\right)-\hat{c}\mu_+\right)\right).\end{aligned}$$

Expanding the numerators of the probabilities by $\pm\hat{c}\mu_+$ and reorganizing according to common factors, we can write $\widehat{X}^n_{k-1}\left(t^n_{k-2}\right)=A^n_{k-1}+B^n_{k-1}$, with

$$\begin{aligned}A^n_{k-1}\equiv&\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\left(\hat{c}\mu^n_{k-1}-\hat{c}\mu_+\right)^2+\frac{\hat{c}\mu_+-\hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\\&\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_++\hat{c}\mu^n_k\right)\left(\hat{c}\mu_+-\hat{c}\mu^n_k\right)+2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_++\mu^n_k\right)-\hat{c}\mu_+\right)\right)\end{aligned}$$

and

$$\begin{aligned}B^n_{k-1}\equiv&\frac{\hat{c}\mu_+-\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\left(\hat{c}\mu^n_{k-1}-\hat{c}\mu_+\right)^2+\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\\&\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_++\hat{c}\mu^n_k\right)\left(\hat{c}\mu_+-\hat{c}\mu^n_k\right)+2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_++\mu^n_k\right)-\hat{c}\mu_+\right)\right).\end{aligned}$$

The indifference condition of type t^n_{k-1}, $\hat{c}\mu^n_k=2\underline{\alpha}t^n_{k-1}-\hat{c}\mu^n_{k-1}$, allows us to substitute for $\hat{c}\mu^n_k$. Hence,

$$\begin{aligned}A^n_{k-1}=&\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\left(\hat{c}\mu^n_{k-1}-\hat{c}\mu_+\right)^2+\frac{\hat{c}\mu_+-\hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\\&\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_++2\underline{\alpha}t^n_{k-1}-\hat{c}\mu^n_{k-1}\right)\left(\hat{c}\mu_+-\left(2\underline{\alpha}t^n_{k-1}-\hat{c}\mu^n_{k-1}\right)\right)\right.\\&\left.+2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t_{k-1}\right]-\hat{c}\mu_+\right)\left(\frac{1}{2-\hat{c}}\left(\hat{c}\mu+2\underline{\alpha}t_{k-1}-\hat{c}\mu^n_{k-1}\right)-\hat{c}\mu_+\right)\right).\end{aligned}$$

Collecting terms with the common factor $\frac{\hat{c}\mathbb{E}\left[\theta|\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\left(\hat{c}\mu^n_{k-1}-\hat{c}\mu_+\right)$ and simplifying, we get

$$\begin{aligned}A^n_{k-1}=&\frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+-\hat{c}\mu^n_{k-1}\right)\left(\hat{c}\mu_++\hat{c}\mu^n_{k-1}\right)+\frac{\hat{c}\mu_+-\hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\\&\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t^n_{k-1}\right)^2+4\underline{\alpha}t^n_{k-1}c\mu_{k-1}\right)+\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+\right)\left(\frac{4}{2-\hat{c}}\left(\underline{\alpha}t^n_{k-1}-\hat{c}\mu^n_{k-1}\right)\right)\right).\end{aligned}$$

Similarly, we can derive

$$\begin{aligned}B^n_{k-1}=&2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]-\hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_++\mu^n_{k-1}\right)-\hat{c}\mu_+\right)+\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}\\&\cdot\left(\frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t^n_{k-1}\right)^2+4\underline{\alpha}t^n_{k-1}\hat{c}\mu^n_{k-1}\right)+\left(\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+\right)\left(\frac{4}{2-\hat{c}}\underline{\alpha}t^n_{k-1}-\frac{4}{2-\hat{c}}\hat{c}\mu^n_{k-1}\right)\right).\end{aligned}$$

We aim at showing that the second lines in A_k and B_k respectively are both positive. We then obtain a lower bound on $\widehat{X}^n_{k-1}$ by discarding them.

Note that

$$\frac{\hat{c}\mu_+-\hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}+\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]-\hat{c}\mu_+}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}=\frac{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]-\hat{c}\mu^n_{k-1}}{\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu^n_{k-1}}.$$

Since $\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]>\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-2}\right]>\mathbb{E}\left[\theta|\,\theta\in\left[t^n_{k-2},t^n_{k-1}\right]\right]=\mu^n_{k-1}$, both the denominator and the numerator are positive.

By the definitions of α, $\underline{\alpha}$, and by convexity, $\hat{c}\mathbb{E}\left[\theta|\,\theta\geq t^n_{k-1}\right]-\hat{c}\mu_+=\hat{c}\alpha\left(t^n_{k-1}\right)t^n_{k-1}\geq\hat{c}\underline{\alpha}t^n_{k-1}$. Moreover, since $a<1$ and $t^n_{k-1}\geq\mu^n_{k-1}$, $\underline{\alpha}t^n_{k-1}-\hat{c}\mu^n_{k-1}=\underline{\alpha}\left(t^n_{k-1}-a\mu^n_{k-1}\right)\geq 0$. Taken together, we get

$$\left(\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t^n_{k-1}\right] - \hat{c}\mu_+\right)\left(\frac{4}{2-\hat{c}}\underline{\alpha}t^n_{k-1} - \frac{4}{2-\hat{c}}\hat{c}\mu^n_{k-1}\right) \geq \hat{c}\underline{\alpha}t^n_{k-1}\left(\frac{4}{2-\hat{c}}\underline{\alpha}t^n_{k-1} - \frac{4}{2-\hat{c}}\hat{c}\mu^n_{k-1}\right),$$

and therefore

$$\begin{aligned}
&\frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t^n_{k-1}\right)^2 + 4\underline{\alpha}t^n_{k-1}\hat{c}\mu^n_{k-1}\right) + \left(\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t^n_{k-1}\right] - \hat{c}\mu_+\right)\left(\frac{4}{2-\hat{c}}\underline{\alpha}t^n_{k-1} - \frac{4}{2-\hat{c}}\hat{c}\mu^n_{k-1}\right)\\
&\geq \frac{\hat{c}}{2-\hat{c}}\left(-4\left(\underline{\alpha}t^n_{k-1}\right)^2 + 4\underline{\alpha}t^n_{k-1}\hat{c}\mu^n_{k-1}\right) + \hat{c}\underline{\alpha}t^n_{k-1}\left(\frac{4}{2-\hat{c}}\underline{\alpha}t^n_{k-1} - \frac{4}{2-\hat{c}}\hat{c}\mu^n_{k-1}\right)\\
&= 0.
\end{aligned}$$

Note that all inequalities involving $\underline{\alpha}$ are strict for the case in which $\alpha\left(t^n_{k-1}\right) > \underline{\alpha}$. This implies that the second lines in A^n_k and B^n_k are indeed positive. Hence, we have

$$\begin{aligned}
X^n_{k-1}\left(t^n_{k-2}\right) &\geq \widehat{X}^n_{k-1}\left(t^n_{k-2}\right)\\
&\geq \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu_+ - \hat{c}\mu^n_{k-1}\right)\left(\hat{c}\mu_+ + \hat{c}\mu^n_{k-1}\right) + 2\left(\hat{c}\mathbb{E}\left[\theta|\,\theta \geq t^n_{k-2}\right] - \hat{c}\mu_+\right)\left(\frac{\hat{c}}{2-\hat{c}}\left(\mu_+ + \mu^n_{k-1}\right) - \hat{c}\mu_+\right).
\end{aligned}$$

This concludes the induction step.

It follows that $X^n_1\left(t^n_0\right) \geq \frac{\hat{c}}{2-\hat{c}}\left(\hat{c}\mu^n_1 + \hat{c}\mu_+\right)\left(\hat{c}\mu_+ - \hat{c}\mu^n_1\right)$.

By definition, $X^n_1\left(t^n_0\right) = \mathbb{E}\left[\left(\hat{c}\mu^n_i - \hat{c}\mu_+\right)^2\right]$. Canceling $\hat{c}$, we get

$$\mathbb{E}\left[\left(\mu^n_i - \mu_+\right)^2\right] \geq \frac{\underline{\alpha}a}{2-\underline{\alpha}a}\left(\mu^2_+ - \left(\mu^n_1\right)^2\right)$$

with strict inequality if $\mathbb{E}[\theta|\,\theta \geq t]$ is strictly convex in t. Decentering again and noting that by the law of iterated expectations $\mathbb{E}\left[\mu^n_i\mu_+\right] = \mathbb{E}\left[(\mu_+)^2\right]$, we can write

$$\mathbb{E}\left[\left(\mu^n_i\right)^2\right] \geq \frac{\underline{\alpha}a}{2-\underline{\alpha}a}\left(\mu^2_+ - \left(\mu^n_1\right)^2\right) + \mu^2_+.$$

Recall that $\phi(0) = \mu_+$ and $\underline{\alpha} = \phi'(0)$. Thus, for limit $n \to \infty$, we have $\mu^n_1 \to 0$ and

$$var(\mu^\infty) = \mathbb{E}\left[\left(\mu^\infty_i\right)^2\right] \geq \frac{\underline{\alpha}a}{2-\underline{\alpha}a}\mu^2_+ + \mu^2_+ = \frac{2}{2-\phi'(0)a}\phi(0)^2. \quad \square$$

Proof of Lemma 2. Straightforward integration gives for any $[\underline{t},\bar{t}] \subseteq \left[0, -\frac{s}{\delta}\right]$,

$$\mathbb{E}\left[\Theta|\Theta \in [\underline{t},\bar{t}]\right] = \frac{s+\bar{t}}{1-\delta} - \frac{1}{1-\delta}\frac{\left(\bar{t}-\underline{t}\right)}{1-\left(\frac{1+\frac{\delta}{s}\bar{t}}{1+\frac{\delta}{s}\underline{t}}\right)^{-\frac{1}{\delta}}}. \tag{14}$$

For the special case of $\bar{t} = -\frac{s}{\delta}$ and $\underline{t} \in \left[0, -\frac{s}{\delta}\right]$, we get

$$\mathbb{E}\left[\Theta|\Theta \geq \underline{t}\right] = \mathbb{E}\left[\Theta|\Theta \geq 0\right] + \frac{1}{1-\delta}\cdot\underline{t} = \frac{s+\underline{t}}{1-\delta}. \tag{15}$$

Hence, the generalized Pareto distribution features linear tail-truncated expectations. Therefore, we can apply the value characterization of Deimen and Szalay (2019), which derives the expected utility of a limit equilibrium given in (4) as an upper bound on the expected utilities of finite equilibria. The variance of μ^n in a Even equilibrium is given by

$$var(\mu^n) = \frac{2}{2-\frac{a}{1-\delta}}\mu^2_+ - \frac{\frac{a}{1-\delta}}{2-\frac{a}{1-\delta}}\left(\mu^n_1\right)^2.$$

The variance of μ^n in an Odd equilibrium is given by

$$var(\mu^n) = \left(1 - \Pr\left[\Theta \in \left[-\frac{a\mu^n_2}{2}, \frac{a\mu^n_2}{2}\right)\right]\right)\cdot\left(\frac{2}{2-\frac{a}{1-\delta}}\mu^2_+ + \frac{\frac{a}{1-\delta}}{2-\frac{a}{1-\delta}}\mu^n_2\mu_+\right).$$

Deimen and Szalay (2019) shows that a limit equilibrium exists for the special case of $\delta = 0$. Here, we extend the proof of existence of a limit equilibrium in Proposition 1 to the class of all logconcave densities, which includes the generalized Pareto distribution with $\delta \in [-1, 0]$. $\square$

Proof of Proposition 5. One can show that our limit equilibrium yields a higher payoff than any finite equilibrium in the communication game. Compare the receiver's expected utility in a limit equilibrium under communication

$$\mathbb{E}u_R(a\mu^\infty,\Theta,a)=a^2\left(var(\mu^\infty)-\sigma^2\right)=a^2\left(\frac{2-\frac{1}{1-\delta}}{2-\frac{a}{1-\delta}}\sigma^2-\sigma^2\right)=-a^2\sigma^2\frac{1-a}{2-a-2\delta}$$

to the receiver's expected utility under delegation $\mathbb{E}u_R(\Theta,\Theta,a)=-(1-a)^2\sigma^2$. The receiver prefers delegation over communication if

$$-(1-a)^2\sigma^2\geq -a^2\sigma^2\frac{1-a}{2-a-2\delta}\quad\Leftrightarrow\quad \delta\geq\frac{2-3a}{2-2a}.\qquad\square$$

Lemma A.7. *Assuming symmetry and logconcavity, the distributions in the generalized Pareto class satisfy Definitions 1 and 2.*

Proof of Lemma A.7. Take two members f,g of the generalized Pareto family with parameters s,δ and s',δ' respectively, such that $s'<s$, $0\geq\delta'>\delta\geq-1$, and $-\frac{s}{\delta}<-\frac{s'}{\delta'}$. Thus, let

$$f_+=\frac{1}{s}\left(1+\frac{\delta}{s}\theta\right)^{-\frac{1}{\delta}-1}\quad\text{and}\quad g_+=\frac{1}{s'}\left(1+\frac{\delta'}{s'}\theta\right)^{-\frac{1}{\delta'}-1}.$$

Consider $\Theta_{f+}\leq_c\Theta_{g+}$: As shown in the proof of Lemma A.8, with $\overline{G}_+:=1-G_+$ and $\overline{F}_+:=1-F_+$, we can equivalently check convexity of $\overline{G}_+^{-1}\overline{F}_+(\theta)$:

We have $\overline{G}_+(\theta)=\left(1+\frac{s'}{\delta'}\theta\right)^{-\frac{1}{\delta'}}$ and $\overline{F}_+(\theta)=\left(1+\frac{s}{\delta}\theta\right)^{-\frac{1}{\delta}}$. Thus $\overline{G}_+^{-1}(u)=\left(u^{-\delta'}-1\right)\frac{\delta'}{s'}$ and

$$\overline{G}_+^{-1}\overline{F}_+(\theta)=\left(\left(1+\frac{s}{\delta}\theta\right)^{\frac{\delta'}{\delta}}-1\right)\frac{\delta'}{s'}.$$

Differentiating twice, we get $\frac{\partial^2}{\partial\theta^2}\overline{G}_+^{-1}\overline{F}_+(\theta)=\left(\frac{\delta'}{\delta}\left(\frac{\delta'}{\delta}-1\right)\left(1+\frac{s}{\delta}\theta\right)^{\frac{\delta'}{\delta}-2}\right)\left(\frac{s}{\delta}\right)^2\frac{\delta'}{s'}>0$, where the inequality follows from noting that $\delta<\delta'<0$ implies that $0<\frac{\delta'}{\delta}<1$.

Consider $\Theta_{f_+}\leq_{uv}\Theta_{g_+}$: Note that $\frac{f_+}{g_+}$ is increasing (decreasing) if $\frac{f'_+}{f_+}>(<)\frac{g'_+}{g_+}$. Noting that

$$\frac{f'_+}{f_+}=\frac{-(1+\delta)\frac{1}{s^2}\left(1+\frac{\delta}{s}\theta\right)^{-\frac{1}{\delta}-2}}{\frac{1}{s}\left(1+\frac{\delta}{s}\theta\right)^{-\frac{1}{\delta}-1}}=-(1+\delta)\frac{1}{s}\left(1+\frac{\delta}{s}\theta\right)^{-1}$$

and

$$\frac{g'_+}{g_+}=-\left(1+\delta'\right)\frac{1}{s'}\left(1+\frac{\delta'}{s'}\theta\right)^{-1},$$

we observe that $\frac{f'_+}{f_+}>\frac{g'_+}{g_+}$ if and only if

$$-(1+\delta)\frac{1}{s}\left(1+\frac{\delta}{s}\theta\right)^{-1}>-\left(1+\delta'\right)\frac{1}{s'}\left(1+\frac{\delta'}{s'}\theta\right)^{-1}.\tag{16}$$

At $\theta=0$, inequality (16) is satisfied since $s'<s$ and $\delta'>\delta$ imply $\left(1+\delta'\right)\frac{1}{s'}>(1+\delta)\frac{1}{s}$.

For $0<\theta<-\frac{s}{\delta}$, inequality (16) is equivalent to

$$\left(1+\delta'\right)\frac{1}{s'}\left(1+\frac{\delta}{s}\theta\right)>(1+\delta)\frac{1}{s}\left(1+\frac{\delta'}{s'}\theta\right).\tag{17}$$

The left side of (17) decreases in θ at rate $\left(1+\delta'\right)\frac{1}{s'}\frac{\delta}{s}$, while the right side decreases at rate $(1+\delta)\frac{1}{s}\frac{\delta'}{s'}$. Now $\delta<\delta'$ implies that $\left(1+\delta'\right)\frac{1}{s'}\frac{\delta}{s}<(1+\delta)\frac{1}{s}\frac{\delta'}{s'}$, so the left side of (17) decreases faster.

Finally, at $\theta=-\frac{s}{\delta}$, the left side of (17) is zero, while the right side is positive, so the inequality is reversed.

It follows that there exists a unique interior mode. $\square$

Proof of Lemma 3. Since the Gaussian distribution features a convex tail-truncated expectation (see Sampford (1953)), the minimal slope for the tail-truncated expectation is obtained at $\theta=0$.

$$\frac{\partial}{\partial t}\mathbb{E}\left[\Theta|\,\Theta\geq t\right]\Big|_{t=0}=\left(\mathbb{E}\left[\Theta|\,\Theta\geq t\right]-t\right)\frac{f(t)}{1-F(t)}\Big|_{t=0}=\frac{\phi(0)}{\sigma}2\frac{1}{\sqrt{2\pi}}.$$

Moreover, we have $\mathbb{E}[\Theta|\Theta \geq t]|_{t=0} = \phi(0) = \sigma \left.\frac{f(t)}{1-F(t)}\right|_{t=0} = \sigma\frac{\sqrt{2}}{\sqrt{\pi}}$. Substituting in (4) for $\phi(0)$ and the minimal slope, we obtain the result. □

Lemma A.8. *Denote the Gaussian distribution by F and the Laplace distribution by G,*
i) then $\Theta_{f+} \leq_c \Theta_{g+}$.
ii) then $\Theta_{f+} \leq_{uv} \Theta_{g+}$.

Proof of Lemma A.8. i) Follows from van Zwet (1964) p.59, as the Gaussian distribution has an increasing hazard rate.

ii) Let g_+ be the Laplace and f_+ be the Gaussian densities truncated to $\theta \geq 0$. Then

$$\frac{f_+(\theta)}{g_+(\theta)} = \frac{\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\theta^2}{2\sigma^2}}}{\frac{\sqrt{2}}{\sigma}e^{-\frac{\sqrt{2}}{\sigma}\theta}} = \frac{e^{\left(\frac{\sqrt{2}}{\sigma}\theta - \frac{\theta^2}{2\sigma^2}\right)}}{2\sqrt{\pi}},$$

and we observe that $\frac{f_+(\theta)}{g_+(\theta)}$ is increasing for low levels of θ and decreasing for high levels of θ. □

Proof of Proposition 6. i) For the Laplace distribution, communication in a limit equilibrium is preferred over delegation if

$$a^2\left(\frac{1}{2-a}\sigma_{\mathcal{L}}^2 - \sigma_{\mathcal{L}}^2\right) \geq -(1-a)^2\sigma_{\mathcal{L}}^2,$$

which holds if and only if $a \leq a_{\mathcal{L}} := \frac{2}{3}$, independently of $\sigma_{\mathcal{L}}^2$.

For the Gaussian distribution, using the lower bound for communication, we obtain that communication in a limit equilibrium is preferred over delegation if

$$a^2\left(\frac{\frac{4}{\pi}}{2-a\frac{2}{\pi}}\sigma_{\mathcal{G}}^2 - \sigma_{\mathcal{G}}^2\right) \geq -(1-a)^2\sigma_{\mathcal{G}}^2,$$

which holds for $a < a_{\mathcal{G}} := 0.702$, independently of $\sigma_{\mathcal{G}}^2$.

As $\frac{1}{2} < a_{\mathcal{L}} < a_{\mathcal{G}}$, the statement follows. In particular, for $a \in \left(\frac{2}{3}, 0.702\right)$ delegation is strictly optimal for a Laplace distribution while communication is strictly optimal for a Gaussian distribution.

ii) Comparing the values of communicating in a limit equilibrium for the Gaussian and Laplace distributions, the Gaussian distribution induces a higher value of communication than the Laplace

$$a^2\left(\frac{\frac{4}{\pi}}{2-a\frac{2}{\pi}}\sigma_{\mathcal{G}}^2 - \sigma_{\mathcal{G}}^2\right) \geq a^2\left(\frac{1}{2-a}\sigma_{\mathcal{L}}^2 - \sigma_{\mathcal{L}}^2\right)$$
$$\Leftrightarrow \quad \frac{\sigma_{\mathcal{G}}^2}{\sigma_{\mathcal{L}}^2}\left(1 - \frac{\frac{4}{\pi}}{2-a\frac{2}{\pi}}\right) \leq \left(1 - \frac{1}{2-a}\right)$$

if $a < 0.858$ and $\sigma_{\mathcal{G}}^2 \leq \sigma_{\mathcal{L}}^2$. □

Proof of Lemma 4. Since the supports are assumed to be $\mathbb{R}$, we have $supp(f) \subseteq supp(g)$. It remains to be shown that the ratio $\frac{f_+(\theta)}{g_+(\theta)}$ is unimodal with mode m an interior maximum.

Logconcavity of the ratio $\frac{f_+(\theta)}{g_+(\theta)}$ is equivalent to $\frac{\partial}{\partial\theta}\left(\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} - \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}\right) \leq 0$. That the difference is falling implies that one of three cases holds: either the difference is positive for all θ, $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} > \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$, negative for all θ, $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} < \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$, or changes sign once, i.e., there is some value m such that $\frac{\frac{\partial}{\partial\theta}f_+(\theta)|_{\theta=m}}{f_+(m)} = \frac{\frac{\partial}{\partial\theta}g_+(\theta)|_{\theta=m}}{g_+(m)}$ and $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} > \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$ for $\theta \in [0, m)$ and $\frac{\frac{\partial}{\partial\theta}f_+(\theta)}{f_+(\theta)} < \frac{\frac{\partial}{\partial\theta}g_+(\theta)}{g_+(\theta)}$ for $\theta \in (m, \overline{S}]$.

The first two cases amount to MLRP on the positive half and can be ruled out by the following argument: In the first case, monotonicity of the likelihood ratio for all $\theta > 0$ implies that $F_+(\theta)$ and $G_+(\theta)$ are ranked in the usual stochastic order, $\Theta_{f+} \geq_{st} \Theta_{g+}$. By symmetry, this implies that $F(\theta)$ and $G(\theta)$ are ordered in the convex order, $\Theta_{f+} \geq_{cx} \Theta_{g+}$. In the second case, both relations are reversed. Both cases imply that the distributions must have different variances, contradicting our assumption.

Hence, case three applies, implying that $\frac{f_+}{g_+}$ is unimodal with unique interior mode m. By concavity the mode is a maximum. □

Proof of Lemma 5. We show that the convex transform order, $\Theta_{f+} \leq_c \Theta_{g+}$, is transitive. Note that

$$G_+^{-1}F_+(\theta) = G_+^{-1}H_+H_+^{-1}F_+(\theta).$$

Since $G_+^{-1}H_+(\theta)$ and $H_+^{-1}F_+(\theta)$ are increasing functions, $G_+^{-1}F_+(\theta)$ is convex if $G_+^{-1}H_+(\theta)$ and $H_+^{-1}F_+(\theta)$ are convex.

Recall that a Laplace distribution is a two-sided exponential distribution. van Zwet (1964) shows that for H_+ the exponential distribution, $H_+^{-1}F_+(\theta)$ is convex for any distribution F_+ with an increasing hazard rate. Since logconcavity of the density implies an increasing hazard rate (Bagnoli and Bergstrom (2005)), $H_+^{-1}F_+(\theta)$ is convex. Likewise, by van Zwet (1964), $H_+^{-1}G_+(\theta)$ is concave for any distribution G_+ with a decreasing hazard rate. Again, logconvexity of the density implies a decreasing hazard rate (Bagnoli and Bergstrom (2005)).

Hence, we need to show that $H_+^{-1}G_+(\theta)$ is concave if and only if $G_+^{-1}H_+(\theta)$ is convex. We note that $H_+^{-1}G_+(\theta)$ is concave if and only if $\frac{g_+\left(G_+^{-1}(u)\right)}{h_+\left(H_+^{-1}(u)\right)}$ is decreasing in $u \in [0,1]$ while $G_+^{-1}H_+(\theta)$ is convex if and only if $\frac{h_+\left(H_+^{-1}(u)\right)}{g_+\left(G_+^{-1}(u)\right)}$ is increasing in $u \in [0,1]$. Hence, $H_+^{-1}F_+(\theta)$ is convex if and only if $H_+^{-1}G_+(\theta)$ is concave. □

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